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A note on Smarandache number related triangles

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Abstract The triangle T(a, b, c) with angles a, b, c, and the triangle $T(a', b', c')$ with angles a', b', c' are said to be pseudo Smarandache related if Z(a) =Z(a′), Z(b) =Z(b′), Z(c) =Z(c′), and the pair of triangles T(a,b, c) and $T(a', b', c')$ are said to be Smarandache related if $S(a) = S(a'), S(b) = S(b'), S(c) = S(c')$, where Z(.) is the pseudo Smarandache function, and S(.) is the Smarandache function. This paper lists all the dissimilar pseudo Smarandache related triangles, under the additional condition that a =a′, found by computer search.

Keywords Pseudo Smarandache function, Smarandache function, pseudo Smarandache related triangles, Smarandache function related triangles.

§1. Introduction

The pseudo Smarandache function, denoted by Z(n), has been introduced by Kashihara [1]. Since then, the pseudo Smarandache function has seen several generalizations in different directions. One such generalization is the concept of the pseudo Smarandache related triangles, proposed by Ashbacher [2]. Actually, the idea of the Smarandache related triangles was introduced by Sastry [3], and Ashcacher [2] extended the idea to include the pseudo Smarandache function as well. The formal definitions of the pseudo Smarandache function, Smarandache function, and the Smarandache number related triangles are given below.

Definition 1.1. For any integer n ≥1, the pseudo Smarandache function, Z(n), is the minimum integer m such that 1+2+· · · +m is divisible by n, that is

$$Z(n) = \min\left\{m : m \in \mathrm{Z}^+, n|\frac{m(m+1)}{2}\right\}, n \geq 1,$$

where Z^+ is the set of all positive integers.

Definition 1.2. Two triangles T(a, b, c) (with angles a, b and c) and $T(a', b', c')$ (with angles a', b' and c'), are said to be Smarandache related if

$$Z(a) = Z(a'), Z(b) = Z(b'), Z(c) = Z(c'),$$

where $a + b + c = 180 = a' + b' + c'$.

Definition 1.3. The Smarandache function S(n) is defined as follows:

$$S(n) = \min\left\{m : m \in \mathrm{Z}^+, n|m!\right\}, n \geq 1,$$

and two triangles T(a, b, c) and $T(a', b', c')$ are said to be Smarandache related if

$$S(a) = S(a'), S(b) = S(b'), S(c) = S(c').$$

Definition 1.4. A triangle is said to be Pythagorean if and only if one of its angle is 90^0.

Definition 1.5. Two triangles are said to be similar if the angles of one triangle are equal to the corresponding angles of the second triangle, in any order.

So far as the authors know, not much work has been done in connection with the Smarandache number related triangles. Ashbacher [2] reports some of the dissimilar pairs of Smarandache number related triangles, found using a computer search.

This paper reports all pairs of dissimilar pseudo Smarandache related triangles, under the additional restriction that a $= a'$. Though such a restricted search might be unwanted, nevertheless, the number of such dissimilar triangles reduces very dramatically, only to 59. On the other hand, the number of Smarandache related triangles under the same condition is 1072. We believe that a closer study of these triangles would be helpful in further research. These triangles are given in the next Section 2. We conclude this paper with some discussion in the final Section 3.

§2. Computational results

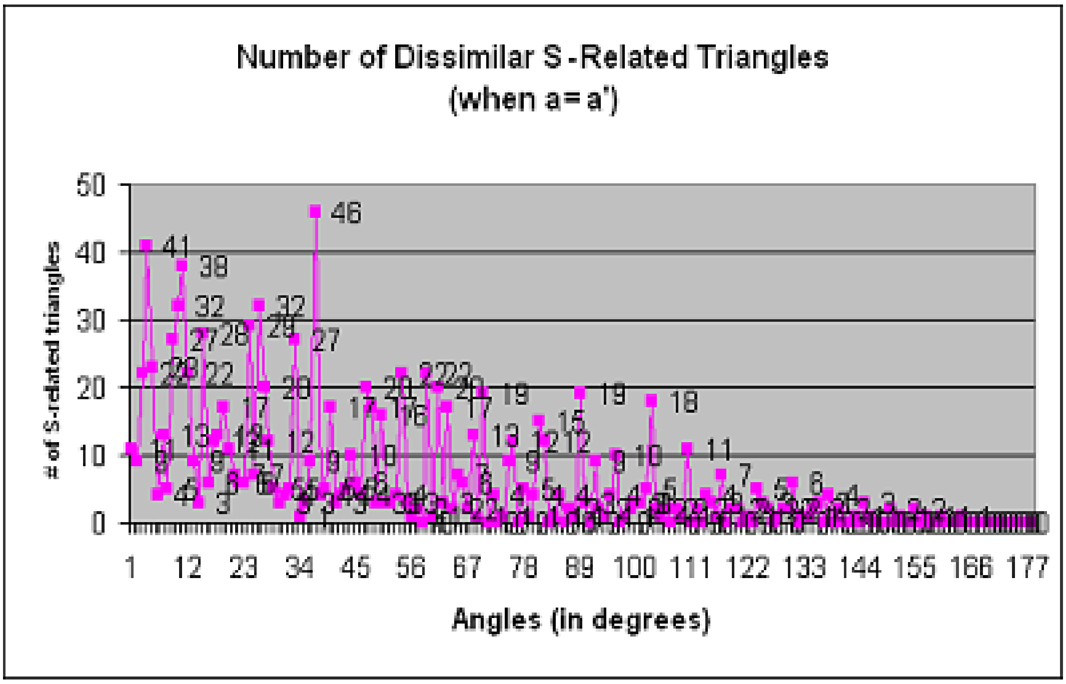
We searched for all dissimilar pseudo Smarandache function related triangles T(a,b,c) (with angles a, b and c) and $T(a', b', c')$, under the additional restriction that a =a′, on a computer. Our findings are given below.

(1) a =a′ =4; T(4,8,168), T(4,120,56) with Z(8) =15 =Z(120), Z(168) =48 =Z(56),
(2) a =a′) =4; T(4,44,132), T(4,88,88) with Z(44) =32 =Z(88) =Z(132),
(3) a =a′ =4; T(4,80,96), T(4,104,72) with Z(80) =64 =Z(104), Z(96) =63 =Z(72),
(4) a =a′ =5; T(5,25,150), T(5,100,75) with Z(25) =24 =Z(100), Z(150) =24 =Z(75),
(5) a =a′ =9; T(9,38,133), T(9,95,76) with Z(38) =19 =Z(95), Z(133) =56 =Z(76),
(6) a =a′ =10; T(10,20,150), T(10,120,50) with Z(20) =15 =Z(120), Z(150) =24 =Z(50),
(7) a =a′ =11; T(11,13,156), T(11,39,130) with Z(13) =12 =Z(39), Z(156) =39 =Z(130),
(8) a =a′ =12; T(12,14,154), T(12,28,140) with Z(14) =7 =Z(28), Z(154) =55 =Z(140),
(9) a =a′ =19; T(19,7,154), T(19,21,140) with Z(7) =6 =Z(21), Z(154) =55 =Z(140),
(10) a =a′ =20; T(20,48,112), T(20,88,72) with Z(48) =32 =Z(88), Z(112) =63 =Z(72),
(11) a =a′ =25; T(25,31,124), T(25,93,62) with Z(31) =30 =Z(93), Z(124) =31 =Z(62),
(12) a =a′ =26; T(26,16,138), T(26,62,92) with Z(16) =31 =Z(62), Z(138) =23 =Z(92),
(13) a =a′ =26; T(26,22,132), T(26,66,88) with Z(22) =11 =Z(66), Z(132) =32 =Z(88),
(14) a =a′ =27; T(27,34,119), T(27,68,85) with Z(34) =16 =Z(68), Z(119) =34 =Z(85),
(15) a =a′ =28; T(28,8,144), T(28,40,112) with Z(8) =15 =Z(40), Z(144) =63 =Z(112),
(16) a =a′ =28; T(28,8,144), T(28,120,32) with Z(8) =15 =Z(120), Z(144) =63 =Z(32),
(17) a =a′ =28; T(28,32,120), T(28,112,40) with Z(32) =63 =Z(112), Z(120) =15 =Z(40),
(18) a =a′ =30; T(30,50,100), T(30,75,75) with Z(50) =24 =Z(75) =Z(100),
(19) a =a′ =32; T(32,37,111), T(32,74,74) with Z(37) =36 =Z(111) =Z(74),
(20) a =a′ =33; T(33,27,120), T(33,117,30) with Z(27) =26 =Z(117), Z(120) =15 =Z(30),

(21) a =a$'$ =36; T(36,32,112), T(36,72,72) with Z(32) =63 =Z(72) =Z(112),
(22) a =a$'$ =37; T(37,11,132), T(37,55,88) with Z(11) =10 =Z(55), Z(132) =32 =Z(88),
(23) a =a$'$ =42; T(42,46,92), T(42,69,69) with Z(46) =23 =Z(69) =Z(92),
(24) a =a$'$ =44; T(44,24,112), T(44,40,96) with Z(24) =15 =Z(40), Z(112) =63 =Z(96),
(25) a =a$'$ =47; T(47,19,114), T(47,57,76) with Z(19) =18 =Z(57), Z(114) =56 =Z(76),
(26) a =a$'$ =48; T(48,20,112), T(48,60,72) with Z(20) =15 =Z(60), Z(112) =63 =Z(72),
(27) a =a$'$ =48; T(48,22,110), T(48,33,99) with Z(22) =11 =Z(33), Z(110) =44 =Z(99),
(28) a =a$'$ =54; T(54,42,84), T(54,70,56) with Z(42) =20 =Z(70), Z(84) =48 =Z(56),
(29) a =a$'$ =55; T(55,25,100), T(55,50,75) with Z(25) =24 =Z(75) =Z(50) =Z(100),
(30) a =a$'$ =60; T(60,8,112), T(60,24,96) with Z(8) =15 =Z(24), Z(112) =63 =Z(96),
(31) a =a$'$ =60; T(60,32,88), T(60,72,48) with Z(32) =63 =Z(72), Z(88) =32 =Z(48),
(32) a =a$'$ =63; T(63,9,108), T(63,36,81) with Z(9) =8 =Z(36), Z(81) =80 =Z(108),
(33) a =a$'$ =64; T(64,12,104), T(64,36,80) with Z(12) =8 =Z(36), Z(104) =64 =Z(80),
(34) a =a$'$ =68; T(68,14,98), T(68,28,84) with Z(14) =7 =Z(28), Z(84) =48 =Z(98),
(35) a =a$'$ =70; T(70,22,88), T(70,66,44) with Z(22) =11 =Z(66), Z(88) =32 =Z(44),
(36) a =a$'$ =72; T(72,4,104), T(72,28,80) with Z(4) =7 =Z(28), Z(104) =64 =Z(80),
(37) a =a$'$ =72; T(72,12,96), T(72,36,72) with Z(12) =8 =Z(36), Z(96) =63 =Z(72),
(38) a =a$'$ =72; T(72,16,92), T(72,62,46) with Z(16) =31 =Z(62), Z(92) =23 =Z(46),
(39) a =a$'$ =72; T(72,20,88), T(72,60,48) with Z(20) =15 =Z(60), Z(88) =32 =Z(48),
(40) a =a$'$ =75; T(75,7,98), T(75,21,84) with Z(7) =6 =Z(21), Z(98) =48 =Z(84),
(41) a =a$'$ =80; T(80,4,96), T(80,28,72) with Z(4) =7 =Z(28), Z(96) =63 =Z(72),
(42) a =a$'$ =80; T(80,25,75), T(80,50,50) with Z(25) =24 =Z(50) =Z(75),
(43) a =a$'$ =81; T(81,11,88), T(81,55,44) with Z(11) =10 =Z(55), Z(88) =32 =Z(44),
(44) a =a$'$ =88; T(88,20,72), T(88,60,32) with Z(20) =15 =Z(60), Z(72) =63 =Z(32),
(45) a =a$'$ =94; T(94,8,78), T(94,60,26) with Z(8) =15 =Z(60), Z(78) =12 =Z(26),
(46) a =a$'$ =100; T(100,20,60), T(100,40,40) with Z(20) =15 =Z(40) =Z(60),
(47) a =a$'$ =103; T(103,11,66), T(103,55,22) with Z(11) =10 =Z(55), Z(66) =11 =Z(22),
(48) a =a$'$ =108; T(108,9,63), T(108,18,54) with Z(9) =8 =Z(18), Z(63) =27 =Z(54),
(49) a =a$'$ =112; T(112,20,48), T(112,24,44) with Z(20) =15 =Z(24), Z(48) =32 =Z(44),
(50) a =a$'$ =120; T(120,20,40), T(120,30,30) with Z(20) =15 =Z(30) =Z(40),
(51) a =a$'$ =128; T(128,13,39), T(128,26,26) with Z(13) =12 =Z(26) =Z(39),
(52) a =a$'$ =130; T(130,2,48), T(130,6,44) with Z(2) =3 =Z(6), Z(48) =32 =Z(44),
(53) a =a$'$ =132; T(132,8,40), T(132,24,24) with Z(8) =15 =Z(24) =Z(40),
(54) a =a$'$ =136; T(136,4,40), T(136,14,30) with Z(4) =7 =Z(14), Z(40) =15 =Z(30),
(55) a =a$'$ =138; T(138,12,30), T(138,18,24) with Z(12) =8 =Z(18), Z(30) =15 =Z(24),
(56) a =a$'$ =140; T(140,4,36), T(140,28,12) with Z(4) =7 =Z(28), Z(36) =8 =Z(12),
(57) a =a$'$ =145; T(145,7,28), T(145,21,14) with Z(7) =6 =Z(21), Z(28) =7 =Z(14),
(58) a =a$'$ =146; T(146,4,30), T(146,14,20) with Z(4) =7 =Z(14), Z(30) =15 =Z(20),
(59) a =a$'$ =154; T(154,2,24), T(154,6,20) with Z(2) =3 =Z(6), Z(24) =15 =Z(20).

We also looked for all dissimilar Smarandache function related triangles T(a,b,c) (with angles a, b and c) and T(a$'$,b$'$,c$'$), under the same condition that a =a$'$, on a computer. Our findings are given below, both in the tabular and graphical forms. For the values of a =a$'$, not

listed in the table, the number of such a pair of triangles is 0 in each case. Thus, for example, there is no pair of dissimilar Smarandache function related triangles each with one angle fixed at 58 degrees.



§3. Some observations and remarks

In Section 2, we report all the pseudo Smarandache function related dissimilar triangles T(a,b,c) and $T(a', b', c')$ (with a+b+c =180 $= a' + b' + c'$), under the additional condition that $a = a'$. The cases that do not appear in the list are either cannot occur or lead to similar triangles.

Ashbacher [2], based on an exhaustive computer search for pairs of all dissimilar pseudo Smarandache function related triangles with values of a in the range 1 ≤a ≤178, reports that a cannot take the following values (1):

1, 15, 23, 35, 41, 45, 51, 59, 65, 67, 71, 73, 77, 79, 82, 83, 86, 87, 89,
90, 91, 97, 101, 102, 105, 107, 109, 113, 115, 116, 118, 121, 123, 125, 126, 127,
131, 134, 135, 137, 139, 141, 142, 143, 148, 149, 151, 152, 153, 157, 158, 159,
161, 163, 164, 166, 167, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178.

The values of a =a$'$ for which we get pairs of dissimilar triangles can be compared with the values given in (1). Ashbacher also gives the values of a in 1 ≤a ≤178 for which there are no dissimilar Smarandache function related pairs of triangles.

a = a.′	Number of S-related triangles	a = a.′	Number of S-related triangles	a = a.′	Number of S-related triangles	a = a.′	Number of S-related triangles	a = a.′	Number of S-related triangles
1	11	28	12	55	3	88	1	122	1
2	9	29	5	56	1	89	19	124	5
3	22	30	3	57	2	90	3	125	3
4	41	31	4	59	22	92	9	126	2
5	23	32	5	60	1	93	2	128	1
6	4	33	27	61	20	94	1	129	2
7	13	34	1	62	3	95	4	130	2
8	5	35	3	63	17	96	10	131	6
9	27	36	9	64	2	98	1	133	1
10	32	37	46	65	7	99	2	134	1
11	38	38	4	66	6	100	3	135	2
12	22	39	5	67	2	101	3	136	3
13	9	40	17	68	13	102	5	138	4
14	3	41	3	69	1	103	18	140	2
15	28	42	4	70	19	104	2	143	1
16	6	43	5	72	4	105	1	145	3
17	12	44	10	74	1	106	2	146	1
18	13	45	6	75	9	108	2	150	2
19	17	46	4	76	12	109	1	152	1
20	11	47	20	78	5	110	11	155	2
21	7	48	17	79	1	112	2	157	1
22	6	49	3	80	4	114	4	160	1
23	6	50	16	81	15	115	3	164	1
24	29	51	3	82	12	116	1		
25	7	52	3	84	1	117	7		
26	32	53	4	85	4	119	2		
27	20	54	22	87	2	120	2		

We get two pairs of 60 degrees triangles, namely, the pairs T(60, 8, 112), T(60, 24, 96) and T(60, 32, 88), T(60, 72, 48), which are pseudo Smarandache function related. There is only one pair of 120 degrees dissimilar triangles, T(120, 20, 40) and T(120, 30, 30), which are pseudo Smarandache function related, while there is no Pythogorean dissimilar pseudo Smarandache function related triangles.

Looking for the pairs of triangles which are Smarandache function related, we get 25. The pairs of triangles which are both Smarandache function related and pseudo Smarandache function related are as follows :

T(4,44,132), T(4,88,88) with S(44) =11 =S(88) =S(132),
T(5,25,150), T(5,100,75) with S(25) =10 =S(100) =S(150) =S(75),
T(9,76,95), T(9,133,38) with S(76) =19 =S(133) =S(95) =S(38),
T(10,20,150), T(10,120,50) with S(20) =5 =S(120), S(150) =10 =S(50),
T(11,13,156), T(11,39,130) with s(13) =13 =S(39) =S(156) =S(130),
T(25,31,124), T(25,93,62) with S(31) =31 =S(93) =Z(124) =S(62),
T(26,22,132), T(26,66,88) with S(22) =11 =S(66) =S(132) =S(88),
T(27,34,119), T(27,68,85) with S(34) =17 =S(68) =S(119) =S(85),
T(30,50,100), T(30,75,75) with S(50) =10 =S(75) =S(100),
T(32,37,111), T(32,74,74) with S(37) =37 =S(111) =S(74),
T(37,11,132), T(37,55,88) with S(11) =11 =S(55) =S(132) =S(88),
T(42,46,92), T(42,69,69) with S(46) =23 =S(69) =S(92),
T(47,19,114), T(47,57,76) with S(19) =19 =S(57) =S(114) =S(76),
T(48,22,110), T(48,33,99) with S(22) =11 =S(33) =S(110) =S(99),
T(54,42,84), T(54,70,56) with S(42) =7 =S(70) =S(84) =S(56),
T(55,25,100), T(55,50,75) with S(25) =10 =S(75) =S(50) =S(100),
T(63,9,108), T(63,36,81) with S(9) =6 =S(36), S(81) =9 =S(108),
T(70,22,88), T(70,66,44) with S(22) =11 =S(66) =S(88) =S(44),
T(80,25,75), T(80,50,50) with S(25) =10 =S(50) =S(75),
T(81,11,88), T(81,55,44) with S(11) =11 =S(55) =S(88) =S(44),
T(100,20,60), T(100,40,40) with S(20) =5 =S(40) =S(60),
T(103,11,66), T(103,55,22) with S(11) =11 =S(55) =S(66) =S(22),
T(120,20,40), T(120,30,30) with S(20) =5 =S(30) =S(40),
T(128,13,39), T(128,26,26) with S(13) =13 =S(26) =S(39),
T(145,7,28), T(145,21,14) with S(7) =7 =S(21) =S(28) =S(14).

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Smarandache's ratio theorem

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Abstract In this paper we present the Smarandache's ratio theorem in the geometry of the triangle.

Keywords Smarandache's ratio theorem, triangle.

§1. The main result

Smarandache's Ratio Theorem

If the points A_1, B_1, C_1 divide the sides $\| BC \| = a, \| CA \| = b$, respectively $\| AB \| = c$ of a triangle $\triangle ABC$ in the same ratio $k > 0$, then

$$\| AA_1 \|^2 + \| BB_1 \|^2 + \| CC_1 \|^2 \geq \frac{3}{4}(a^2 + b^2 + c^2).$$

Proof. Suppose $k > 0$ because we work with distances.

$$\| BA_1 \| = k \| BC \|, \| CB_1 \| = k \| CA \|, \| AC_1 \| = k \| AB \| .$$

We'll apply tree times Stewart's theorem in the triangle $\triangle ABC$, with the segments AA_1, BB_1, respectively CC_1:

$$\| AB \|^2 \cdot \| BC \| (1-k) + \| AC \|^2 \cdot \| BC \| k - \| AA_1 \|^2 \cdot \| BC \| = \| BC \|^3 (1-k)k,$$

where

$$\| AA_1 \|^2 = (1-k) \| AB \|^2 + k \| AC \|^2 - (1-k)k \| BC \|^2 .$$

Similarly,

$$\| BB_1 \|^2 = (1-k) \| BC \|^2 + k \| BA \|^2 - (1-k)k \| AC \|^2 .$$

$$\| CC_1 \|^2 = (1-k) \| CA \|^2 + k \| CB \|^2 - (1-k)k \| AB \|^2 .$$

By adding these three equalities we obtain:

$$\| AA_1 \|^2 + \| BB_1 \|^2 + \| CC_1 \|^2 = (k^2 - k + 1)(\| AB \|^2 + \| BC \|^2 + \| CA \|^2),$$

which takes the minimum value when $k = \frac{1}{2}$, which is the case when the three lines from the enouncement are the medians of the triangle.

The minimum is $\frac{3}{4}(\| AB \|^2 + \| BC \|^2 + \| CA \|^2)$.

§2. Open problems on Smarandache's ratio theorem

1. If the points $A'_1, A'_2, \ldots, A'_n$ divide the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ of a polygon in a ratio $k > 0$, determine the minimum of the expression:

$$\| A_1A'_1 \|^2 + \| A_2A'_2 \|^2 + \cdots + \| A_nA'_n \|^2 .$$

2. Similarly question if the points $A'_1, A'_2, \ldots, A'_n$ divide the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ in the positive ratios $k_1, k_2, \ldots, k_n$ respectively.

3. Generalize this problem for polyhedrons.

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Generalizations on poly-Eulerian numbers and polynomials

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Abstract The concept of Euler numbers H_n, and their generalization to $H_n(u)$ where u is a real algebraic number, was studied by Frobenius. The generalized n-th Euler number $H_n^{(k)}(u)$ which is called poly-Eulerian number will be generalized to $H_n^{(k)}(u,a,b)$. The corresponding polynomial $H_n^{(k)}(u,a,b)$ will then be introduced by $H_n^{(k)}(u,x,a,b,c)$, the so called generalized poly-Eulerian polynomials which depends on four positive real parameters. Some properties of these polynomials and their relations will be established.

Keywords Bernoulli polynomials, Euler polynomials, generalized poly-Eulerian polynomials, stirling numbers of the second kind, generating functions.

§1. Introduction

In the 17th century a topic of mathematical interest was finite sums of powers of integers such as the series $1+2+3+\cdots+(n-1)$ or the series $1^2+2^2+3^2+\cdots+(n-1)^2$. The closed form for these finite sums were known, but the sum of the more general case $1^k+2^k+3^k+\cdots+(n-1)^k$ was not. It was the mathematician Jacob Bernoulli who solved this problem. After introducing the Bernoulli numbers, Euler introduced the Euler numbers to study the sum $T_k(n)=\sum_{r=0}^{n-1}(-1)^r r^k$. The Bernoulli and Euler arise in Taylor series in the expansions:

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!} \text{ and } \frac{2}{e^x+1}=\sum_{k=0}^{\infty}E_k\frac{x^k}{k!}.$$

Respectively, see [5], [9] and [11]-[26].

Bernoulli proved the following statement:

$$\sum_{k=1}^{n-1}k^p=\sum_{k=1}^{n-1}\frac{B_k}{k!}\frac{p!}{(p+1-k)!}n^{p+1-k}.$$

In 2002, Q. M. Luo, L. Debnath and F. Qi in [11] and [13]-[15], defined the generalization of Bernoulli polynomials as

$$\frac{tc^{xt}}{b^t-a^t}=\sum_{n=0}^{\infty}\frac{B_n(x,a,b,c)}{n!}t^n \quad |t|<\frac{2\pi}{|\ln b-\ln a|}.$$

They showed that

$$x^n=\frac{1}{(n+1)(\ln b)^n}\left[B_{n+1}(x+1,1,b,b)-B_{n+1}(x,1,b,b)\right].$$

Kaneko in [6] introduced and studied poly-Bernoulli numbers which generalized the classical Bernoulli numbers.

The poly-Bernoulli numbers $B_n^{(k)}$ with $k \in Z$, $n \in N$ are defined by the following generating series.

$$\frac{Li_k\left(1-e^{-x}\right)}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!},$$

where, for any integer k, $Li_k\left(z\right)$ denotes the formal power series (for k-th polylogarithm if $k \geq 1$ and a rational function if $k \leq 1$) $\sum_{m=1}^{\infty} \frac{z^m}{m^k}$, $|z| < 1$. (see [2], [7], [9] and [27].)

So we have

$$Li_1\left(z\right) = -\log\left(1-z\right), \; Li_0\left(z\right) = \frac{z}{1-z}, \; Li_{-1}\left(z\right) = \frac{z}{\left(1-z\right)^2}, \; Li_{-2}\left(z\right) = \frac{z(z+1)}{\left(1-z\right)^3}, \; \cdots$$

We have an explicit formula for $Li_{-n}\left(z\right)$:

$$Li_{-n}\left(z\right) = \sum_{k=1}^{n+1} \frac{\left(-1\right)^{n+k+1}\left(k-1\right)!S\left(n+1,k\right)}{\left(1-z\right)^k},$$

where $S\left(n,k\right)$ are stirling numbers of second kind.

Later in 1998 Jin-Woo introduced poly-Eulerian numbers. In this paper we introduce the generalized poly-Eulerian polynomials and from this, we investigate the classical relationship involving generalized poly-Eulerian polynomials and Bernoulli polynomials.

§2. Introduction and preliminary concepts

In the present paper we shall develop a number of generalizations of the poly-Eulerian numbers and polynomials.

Definition 1. For algebraic real number u, the generalized Euler numbers $H_m\left(u\right)$ are defined by

$$\frac{1-u}{e^t-u} = \sum_{m=0}^{\infty} \frac{H_m\left(u\right)}{m!} t^m .$$

Thus we have the relation

$$H_0\left(u\right) = 1 \, , \; uH_k\left(u\right) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array}\right) H_j\left(u\right),$$

and

$$H_k\left(u\right) = \frac{1}{u-1} \sum_{j=0}^{k-1} \left(\begin{array}{c} k \\ j \end{array}\right) H_j\left(u\right), \text{ for } \; u \neq 1 \; .$$

(see [1].)

Definition 2. The k-th polylogarithm is defined by

$$Li_k\left(z\right) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}, \text{ for } \; k \geq 1 \; , \; |z| < 1 \; .$$

(see [7] and [17].)

Definition 3. The poly-Euler numbers $H_n^{(k)}(u)$ is defined by

$$\frac{Li_k\left(1-e^{(1-u)}\right)}{u-e^t}=\sum_{n=0}^{\infty}H_n^{(k)}\frac{t^n}{n!},\ \text{for}\ \ k\geq 1\ .$$

(see [1].)

The left hand side of this equation can be written as

$$\frac{1}{u-e^x}\int_1^u\frac{1}{e^{-(1-t)}-1}\int_1^t\frac{1}{e^{-(1-t)}-1}\ \cdots\int_1^u\frac{t-1}{e^{-(1-t)}-1}dtdt\cdots dt=\sum_{n=0}^{\infty}H_n^{(k)}(u)\,\frac{x^n}{n!},$$

where u is a real number.

Definition 4. The poly-Eulerian polynomials are defined by

$$\frac{Li_k\left(1-e^{(1-u)}\right)}{u-e^t}e^{xt}=\sum_{n=0}^{\infty}\frac{H_n^{(k)}(u\,;\,x)}{n!}t^n\ .$$

By definition 4, we get

$$\left(H_n^{(k)}(u)+x\right)^n=H_n^{(k)}(u\,;\,x)\quad\forall n\geq 0\,,\ k\geq 1\ .$$

Definition 5. Let $a,b>0$. The generalized poly-Eulerian numbers $H_n^{(k)}(u\,;\,a,b)$ are defined by

$$\frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}=\sum_{n=0}^{\infty}\frac{H_n^{(k)}(u\,;\,a,b)}{n!}t^n\ .$$

Definition 6. Let $a,b>0$. We define the generalized poly-Eulerian polynomials by

$$\frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}e^{xt}=\sum_{n=0}^{\infty}\frac{H_n^{(k)}(u\,;x,\,a,b)}{n!}\ .$$

Definition 7. The generalization of $H_n^{(k)}(u;x,a,b)$ is $H_n^{(k)}(u;x,a,b,c)$ which is defined by

$$\frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{xt}=\sum_{n=0}^{\infty}\frac{H_n^{(k)}(u\,;x,\,a,b,c)}{n!}t^n\ .$$

§3. Main results

Theorem 1. For $a,b>0$, and algebraic real number u we have

$$H_n^{(k)}(u\,;\,a,b)=H_n^{(k)}\left(u,\frac{\ln a}{\ln a+\ln b}\right)(\ln a+\ln b)^n\ .$$

Proof of Theorem 1.

$$\begin{aligned}
\frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t} &= \sum_{n=0}^{\infty}\frac{H_n^{(k)}\left(u\,;\,a,b\right)}{n!}t^n\\
&= \frac{1}{a^{-t}}\frac{Li_k\left(1-e^{(1-u)}\right)}{u-(ab)^t}\\
&= e^{t\ln a}\frac{Li_k\left(1-e^{(1-u)}\right)}{u-e^{t(\ln a+\ln b)}}\\
&= \sum_{n=0}^{\infty}H_n^{(k)}\left(u;\frac{\ln a}{\ln a+\ln b}\right)(\ln a+\ln b)^n\frac{t^n}{n!},
\end{aligned}$$

so by evaluating the coefficients we obtain:

$$H_n^{(k)}\left(u\,;\,a,b\right)=H_n^{(k)}\left(u,\frac{\ln a}{\ln a+\ln b}\right)(\ln a+\ln b)^n.$$

Theorem 2. For $a,b>0$ and algebraic real number u we have

$$H_n^{(k)}(u\,;a,b)=\sum_{i=0}^{n}(\ln a+\ln b)^i(\ln a)^{n-i}\binom{n}{i}H_i^{(k)}(u)\ .$$

Proof of Theorem 2. We Have

$$\begin{aligned}
\frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t} &= a^t\frac{Li_k\left(1-e^{(1-u)}\right)}{u-(ab)^t}\\
&= \left(\sum_{k=0}^{\infty}\frac{(\ln a)^k}{k!}t^k\right)\left(\sum_{n=0}^{\infty}\frac{(\ln a+\ln b)^n}{n!}H_n^{(k)}(u)\,t^n\right)\\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}H_i^{(k)}(u)\,\frac{(\ln a+\ln b)^i}{i!\,(n-i)!}(\ln a)^{n-i}\right)x^n,
\end{aligned}$$

so by evaluating the coefficients we have:

$$H_n^{(k)}(u\,;a,b)=\sum_{i=0}^{n}(\ln a+\ln b)^i(\ln a)^{n-i}\binom{n}{i}H_i^{(k)}(u)\,.$$

Proposition 1. We have

$$H_n^{(k)}(u)=-\,H_n(u)\sum_{l=1}^{\infty}\frac{(1-u)^{l-1}}{l!}\sum_{m=0}^{l-1}\frac{(-1)^{m+1}(m+1)!}{(m+1)^k}S(l,m+1)\,,\ \text{for}\ n\geq 0\,,\ k\geq 1\,.$$

Proof of Proposition 1. According to the preceding definitions, the desired result is proved obviously.

Corollary 1. For real algebraic $u,n\geq 0$, and $k\geq 1$ we have:

$$H_n^{(k)}(u;\,a,b)=\sum_{i=0}^{n}\sum_{l=1}^{\infty}\sum_{m=0}^{l-1}(-1)^{m+1}(\ln a+\ln b)^i(-H_i(u))\binom{n}{i}\times$$

$$(\ln a)^{n-i} \frac{(1-u)^{l-1}}{l!} \frac{(m+1)!}{(m+1)^k} S(l\,,\,m+1)\,,$$

where $S(n\,,\,m)$ with $(n \geq 0\,,\,0 \leq m \leq n)$ is the Stirling numbers of the second kind.

Proof of Corollary 1. By theorem 2 and proposition 1, the proof is obvious.

Theorem 3. Let $a, b, c > 0$ and u be a real algebraic number. For $x \in \mathbb{R}$ and $n \geq 0$

$$H_n^{(l)}(u; a, b, c) = \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} H_k^{(l)}(u; a, b)\, x^{n-k}\,.$$

Proof of Theorem 3. We have

$$\begin{aligned}
\frac{Li_l(1-e^{(1-u)})}{ua^{-t}-b^t} c^{xt} &= \sum_{n=0}^{\infty} H_n^{l)}(u; x, a, b, c) \frac{t^n}{n!} \\
&= \left(\sum_{k=0}^{\infty} H_k^{(l)}(u; a, b) \frac{t^n}{n!} \right) \left(\sum_{i=0}^{\infty} \frac{(\ln c)^i x^i}{i!} t^i \right) \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{(\ln c)^{k-i}}{i!(k-i)!} H_k^{(l)}(u; a, b) x^{k-i} t^k \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} H_k^{(l)}(u; a, b)\, x^{n-k} \right) \frac{t^n}{n!}.
\end{aligned}$$

Therefore the proof is complete.

Theorem 4. Let $a, b, c > 0$ and $x \in \mathbb{R}$, $n \geq 0$ we have

$$H_n^{(l)}(u; x, a, b, c) = \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} H_k^{(l)} \left(u; \frac{\ln a}{\ln a + \ln b} \right) (\ln a + \ln b)^k x^{n-k}\,.$$

Proof of Theorem 4. We have

$$H_n^{(l)}(u; x, a, b, c) = \sum_{k=0}^{n} \binom{n}{k} (\ln c)^{n-k} H_k^{(l)}(u; a, b)\, x^{n-k},$$

and

$$H_n^{(l)}(u\,;\,a, b) = H_n^{(l)} \left(u, \frac{\ln a}{\ln a + \ln b} \right) (\ln a + \ln b)^n$$

and if we combine these two formulas, proof will be completed.

Theorem 5. Let $a, b, c > 0$ and $x \in \mathbb{R}$, $n \geq 0$ then we have

$$H_n^{(l)}(u; x,\, a, b, c) = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{k}{j} (\ln c)^{n-k} (\ln a)^{k-j} (\ln a + \ln b)^k H_j^{(l)}(u)\, x^{n-k}.$$

Proof of Theorem 5. Proof is obvious due to theorem 4.

Theorem 6. Let $a, b, c > 0$ and $x \in \mathbb{R}$, $n \geq 0$ then

$$H_n^{(k)}(u\,;\,1-x, a, b, c) = H_n^{(k)} \left(u, -x, ac, \frac{b}{c}, c \right)\,.$$

Proof of Theorem 6. We have

$$\begin{aligned}\frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{(1-x)t} &= \frac{Li_k\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{-xt}\cdot c^t\\ &= \frac{Li_k\left(1-e^{(1-u)}\right)}{u\left(ac\right)^{-t}-\left(\frac{b}{c}\right)^t}c^{-xt}\\ &= \sum_{n=0}^{\infty}H_n^{(k)}\left(u,-x,ac,\frac{b}{c},c\right)\frac{t^n}{n!},\end{aligned}$$

so by evaluating the coefficients we have:

$$H_n^{(k)}\left(u\,;\,1-x,a,b,c\right)=H_n^{(k)}\left(u,-x,ac,\frac{b}{c},c\right).$$

Theorem 7. Let $a,b,c>0$ and $x,y\in\mathbb{R}$, $n\geq 0$ and u be a real algebraic number, then

$$\begin{aligned}H_n^{(l)}\left(u,x+y,a,b,c\right) &= \sum_{k=0}^{n}\binom{n}{k}\left(\ln c\right)^{n-k}H_n^{(l)}\left(u;x,a,b,c\right)y^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}\left(\ln c\right)^{n-k}H_n^{(l)}\left(u,y,a,b,c\right)x^{n-k}.\end{aligned}$$

Proof of Theorem 7. We have

$$\begin{aligned}\frac{Li_l\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{(x+y)t} &= \sum_{n=0}^{\infty}H_n^{(l)}\left(u,x+y,a,b,c\right)\frac{t^n}{n!}\\ &= \frac{Li_l\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{xt}c^{yt}\\ &= \left(\sum_{n=0}^{\infty}H_n^{(l)}\left(u,x,a,b,c\right)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{\infty}\frac{y^i\left(\ln c\right)^i}{i!}t^i\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}y^{n-k}\left(\ln c\right)^{n-k}H_n^{(l)}\left(u,x,a,b,c\right)\right)\frac{t^n}{n!}.\end{aligned}$$

Hence we also have

$$\begin{aligned}\frac{Li_l\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{(x+y)t} &= \frac{Li_l\left(1-e^{(1-u)}\right)}{ua^{-t}-b^t}c^{yt}c^{xt}\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}\left(\ln c\right)^{n-k}H_n^{(l)}\left(u,y,a,b,c\right)\right)\frac{t^n}{n!}.\end{aligned}$$

So the proof is complete.

Now we give some results about derivatives and integrals of the generalized poly-Euler polynomial $H_n^{(k)}\left(u,x,a,b,c\right)$ as follows.

Theorem 8. Let $a,b,c>0$ and u be a real algebraic number. For $x\in\mathbb{R}$, $n\geq 0$ and For any nonnegative integer l and real numbers α and β we have

$$\frac{\partial^l H_n^{(k)}\left(u;x,a,b,c\right)}{\partial x^l}=\frac{n!}{(n-l)!}\left(\ln c\right)^l H_{n-l}^{(k)}\left(u;x,a,b,c\right),$$

$$\int_{\alpha}^{\beta} H_n^{(k)}(u;t,a,b,c)\,dt = \frac{1}{(n+1)\ln c}\left[H_{n+1}^{(k)}(u;\beta,a,b,c) - H_{n+1}^{(k)}(u;\alpha,a,b,c)\right].$$

Here ${\partial^l}/{\partial x^l}$ denote the l-th derivative with respect to x.

Proof of Theorem 8. By induction on l, the proof is complete.

Corollary 2. $H_n^{(k)}(u;x) = H_n^{(k)}\left(u;e^x,e^{1-x}\right)$.

Proof of Corollary 2. By definition 1.5 we have

$$\begin{aligned}\frac{Li_k\left(1-e^{(1-u)}\right)}{u-e^t}e^{xt} &= \sum_{n=0}^{\infty} H_n^{(k)}(u,x)\frac{t^n}{n!} = \frac{Li_k\left(1-e^{(1-u)}\right)}{ue^{-xt}-e^{t-xt}} \\ &= \frac{Li_k\left(1-e^{(1-u)}\right)}{u\left(e^x\right)^{-t}-\left(e^{1-x}\right)^t} = \sum_{n=0}^{\infty} H_n^{(k)}\left(u,e^x,e^{1-x}\right)\frac{t^n}{n!}.\end{aligned}$$

So proof is complete.

Theorem 9. Let $a,b,c>0$ and u be a real algebraic number. For $x\in\mathbb{R}$ and $n\geq 0$ we have

$$H_n^{(k)}(u;x,a,b,c) = H_n^{(k)}\left(u\,,\frac{\ln a + x\ln c}{\ln a+\ln b}\right).$$

Proof of Theorem 9. We have:

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^{(k)}(u,x,a,b,c)\frac{t^n}{n!} &= \frac{Li_k(1-e^{(1-u)})}{ua^{-t}-b^t}c^{xt} \\ &= \frac{1}{a^{-t}}\frac{Li_k(1-e^{(1-u)})}{u-(ab)^t}c^{xt} \\ &= a^t c^{xt}\frac{Li_k(1-e^{(1-u)})}{u-e^{t(\ln a+\ln b)}}e^{t\ln a}e^{xt\ln c}\frac{Li_k(1-e^{(1-u})}{u-e^{t(\ln a+\ln b)}} \\ &= e^{t(\ln a+x\ln c)}\frac{Li_k(1-e^{(1-u)})}{u-e^{t(\ln a+\ln b)}} \\ &= \sum_{n=0}^{\infty} H_n^{(k)}(u\,;\frac{\ln a+x\ln c}{\ln a+\ln c})\frac{t^n}{n!}.\end{aligned}$$

So $H_n^{(k)}(u;x,a,b,c) = H_n^{(k)}\left(u\,,\dfrac{\ln a+x\ln c}{\ln a+\ln b}\right)$ and the proof is complete.

GI-Sang Cheon in [10] investigated the classical relationship involving Bernoulli and Euler polynomials. Now we want to consider the relationship involving generalized poly-Eulerian and Bernoulli polynomials

Theorem 10. For positive number b, the following identity holds between the generalized poly-Eulerian polynomials and the Bernoulli polynomials:

$$\begin{aligned}H_n^{(l)}(u,x+y,1,b,b) &= \sum_{k=0}^{n}\begin{pmatrix} n \\ k\end{pmatrix}\frac{1}{n-k+1}\left[H_{k+1}^{(l)}(u,y+1,1,b,b)\right. \\ &\quad \left.-H_{k+1}^{(l)}(u,y,1,b,b)\right]B_{n-k}(x,1,b,b).\end{aligned}$$

Proof of Theorem 10. By the following identity from theorem 6 we have:

$$H_n^{(l)}(u,x+y,1,b,b) = \sum_{k=0}^{n}(\ln b)^{n-k} H_k^{(l)}(u,y,1,b,b)\,x^{n-k},$$

and

$$x^{n-k}=\frac{1}{(n-k+1)(\ln b)^{n-k}}\left[B_{n-k+1}(x+1,1,b,b)-B_{n-k+1}(x,1,b,b)\right].$$

We obtain

$$H_{n}^{(l)}(u,x+y,1,b,b)$$

$$\begin{aligned}
&= \sum_{k=0}^{n}\frac{1}{(n-k+1)}\begin{pmatrix} n \\ k \end{pmatrix}H_{k}^{(l)}(u,y,1,b,b)\left[B_{n-k+1}(x+1,1,b,b)-B_{n-k+1}(x,1,b,b)\right]\\
&= \sum_{k=0}^{n}\frac{1}{(n-k+1)}\begin{pmatrix} n \\ k \end{pmatrix}H_{k}^{(l)}(u,y,1,b,b)\left[\sum_{j=0}^{n-k+1}\begin{pmatrix} n-k+1 \\ j \end{pmatrix}B_{j}(x,1,b,b)\right.\\
&\quad \left.-B_{n-k+1}(x,1,b,b)\right]\\
&= \sum_{j=0}^{n}\frac{1}{(n-j+1)}\begin{pmatrix} n \\ j \end{pmatrix}B_{j}(x,1,b,b)\sum_{k=0}^{n-j+1}\begin{pmatrix} n-j+1 \\ k \end{pmatrix}H_{k}^{(l)}(u,y,1,b,b)\\
&\quad -\sum_{k=0}^{n}\frac{1}{(n-k+1)}\begin{pmatrix} n \\ k \end{pmatrix}H_{k}^{(l)}(u,y,1,b,b)B_{n-k+1}(x,1,b,b)\\
&= \sum_{j=0}^{n}\frac{1}{(n-j+1)}\begin{pmatrix} n \\ j \end{pmatrix}H_{n-j+1}^{(l)}(u,y+1,1,b,b)B_{j}(x,1,b,b)\\
&\quad -\sum_{k=0}^{n}\frac{1}{(n-k+1)}\begin{pmatrix} n \\ k \end{pmatrix}H_{k}^{(l)}(u,y,1,b,b)B_{n-k+1}(x,1,b,b)\\
&= \sum_{k=0}^{n}\frac{1}{(n-k+1)}\begin{pmatrix} n \\ k \end{pmatrix}\left[H_{k+1}^{(l)}(u,y+1,1,b,b)-H_{k+1}^{(l)}(u,y,1,b,b)\right]B_{n-k}(x,1,b,b).
\end{aligned}$$

Theorem 11. (GI-Sang Cheon) We have

$$B_{n}(x)=\sum_{k=0,k\neq 1}^{n}\begin{pmatrix} n \\ k \end{pmatrix}B_{k}E_{n-k}(x)\ .$$

(see [10].)

Now we have an analogous formula for $E_n(x)$.

Theorem 12. If we set $l=1$, $u=-1$ and $b=e$ in theorem 10 we obtain

$$E_{n}(x)=-\sum_{k=0}^{n}\frac{2}{k+1}\begin{pmatrix} n \\ k \end{pmatrix}E_{k+1}B_{n-k}(x)\ .$$

(see [8].)

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Menelaus's theorem for hyperbolic quadrilaterals in the Einstein relativistic velocity model of hyperbolic geometry

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Abstract In this study, we present (i) a proof of the Menelaus theorem for quadrilaterals in hyperbolic geometry, (ii) and a proof for the transversal theorem for triangles, and (iii) the Menelaus's theorem for n-gons.

Keywords Hyperbolic geometry, hyperbolic triangle, hyperbolic quadrilateral, Menelaus theorem, transversal theorem, gyrovector.

§1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. Here, in this study, we present a proof of Menelaus's theorem for quadrilaterals, a proof for the transversal theorem, and a proof of Menelaus's theorem for n-gons in the Einstein relativistic velocity model of hyperbolic geometry. The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC such that l meets BC in D, CA in E, and AB in F, then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [1]. F. Smarandache (1983) has generalized the Theorem of Menelaus for any polygon with $n \geq 4$ sides as follows: If a line l intersects the n-gon $A_1A_2 \ldots A_n$ sides $A_1A_2, A_2A_3, \ldots$, and A_nA_1 respectively in the points $M_1, M_2, \cdots$, and M_n, then $\frac{M_1A_1}{M_1A_2} \cdot \frac{M_2A_2}{M_2A_3} \cdot \ldots \cdot \frac{M_nA_n}{M_nA_1} = 1$ [2].

Let D denote the complex unit disc in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + \bar{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$.

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$.

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$.

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$.

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$.

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$.

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of onedimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$.

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$.

Definition 1. Let ABC be a gyrotriangle with sides a, b, c in an Einstein gyrovector space $(V_s, \oplus, \otimes)$, and let h_a, h_b, h_c be three altitudes of ABC drawn from vertices A, B, C perpendicular to their opposite sides a, b, c or their extension, respectively. The number

$$S_{ABC} = \gamma_a a \gamma_{h_a} h_a = \gamma_b b \gamma_{h_b} h_b = \gamma_c c \gamma_{h_c} h_c$$

is called the gyrotriangle constant of gyrotriangle ABC (here $\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{s^2}}}$ is the gamma factor). (see [3, pp558])

Theorem 1. (The Gyrotriangle Constant Principle) Let A_1BC and A_2BC be two gyrotriangles in a Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$, $A_1 \neq A_2$ such that the two gyrosegments A_1A_2 and BC, or their extensions, intersect at a point $P \in \mathbb{R}^2_s$, as shown in Figs 1-2. Then,

$$\frac{\gamma_{|A_1P|} |A_1P|}{\gamma_{|A_2P|} |A_2P|} = \frac{S_{A_1BC}}{S_{A_2BC}}.$$

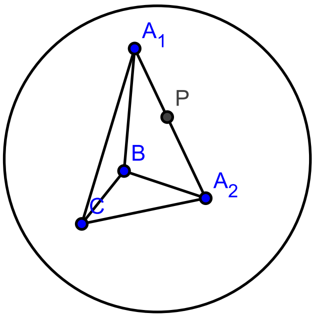

Figure 1

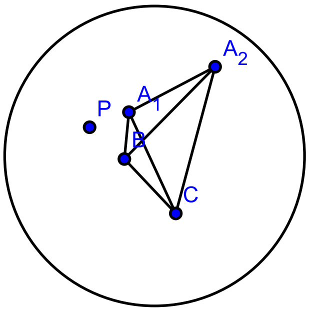

Figure 2

([3, pp 563])

Theorem 2. (The Hyperbolic Theorem of Menelaus in Einstein Gyrovector Space) Let $\mathbf{a}_1, \mathbf{a}_2$, and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(V_s, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ at points $\mathbf{a}_{12}, \mathbf{a}_{13}, \mathbf{a}_{23,}$ as in Figure 3, then

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\,\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\,\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\,\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|}=1.$$

(see [3, pp 463])

For further details we refer to the recent book of A.Ungar [3].

§2. Menelaus's theorem for hyperbolic quadrilaterals

In this section, we prove Menelaus's theorem for hyperbolic quadrilateral.

Theorem 3. If l is a gyroline not through any vertex of a gyroquadrilateral $ABCD$ such that l meets AB in X, BC in Y, CD in Z, and DA in W, then

$$\frac{\gamma_{|AX|}|AX|}{\gamma_{|BX|}|BX|}\cdot\frac{\gamma_{|BY|}|BY|}{\gamma_{|CY|}|CY|}\cdot\frac{\gamma_{|CZ|}|CZ|}{\gamma_{|DZ|}|DZ|}\cdot\frac{\gamma_{|DW|}|DW|}{\gamma_{|AW|}|AW|}=1. \tag{1}$$

Proof of Theorem 3. Let T be the intersection point of the gyroline DB and the gyroline XYZ (See Figure 4). If we use a Theorem 2 in the triangles ABD and BCD respectively, then

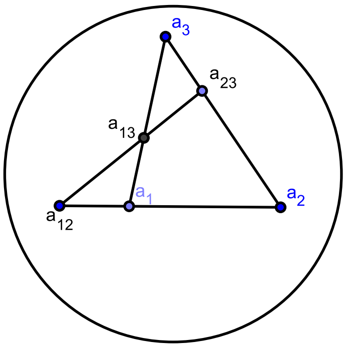

Figure 3

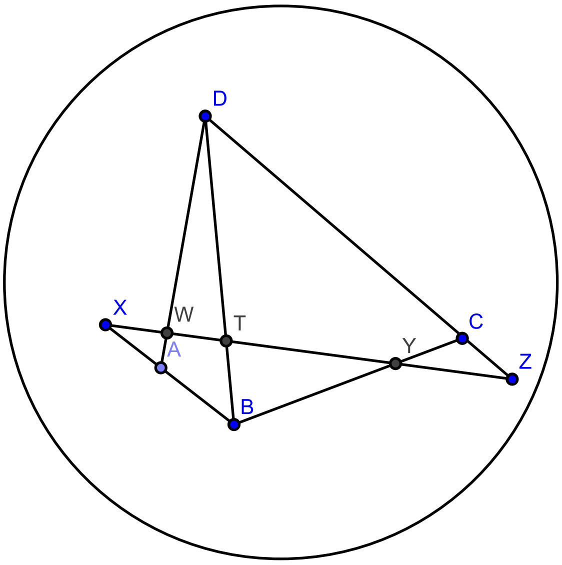

Figure 4

$$\frac{\gamma_{|AX|}|AX|}{\gamma_{|BX|}|BX|}\cdot\frac{\gamma_{|BT|}|BT|}{\gamma_{|DT|}|DT|}\cdot\frac{\gamma_{|DW|}|DW|}{\gamma_{|AW|}|AW|}=1, \tag{2}$$

and

$$\frac{\gamma_{|DT|}|DT|}{\gamma_{|BT|}|BT|}\cdot\frac{\gamma_{|CZ|}|CZ|}{\gamma_{|DZ|}|DZ|}\cdot\frac{\gamma_{|BY|}|BY|}{\gamma_{|CY|}|CY|}=1. \tag{3}$$

Multiplying relations (2) and (3) member with member, we obtain the conclusion.

§3. The hyperbolic transversal theorem for triangles

In this section, we prove the hyperbolic transversal theorem for triangles.

Theorem 4. Let D be on gyroside BC, and l is a gyroline not through any vertex of a

gyrotriangle ABC such that l meets AB in M, AC in N, and AD in P, then

$$\frac{\gamma_{|AM|}|AM|}{\gamma_{|AB|}|AB|} \cdot \frac{\gamma_{|AC|}|AC|}{\gamma_{|AN|}|AN|} \cdot \frac{\gamma_{|PN|}|PN|}{\gamma_{|PM|}|PM|} \cdot \frac{\gamma_{|DB|}|DB|}{\gamma_{|DC|}|DC|} = 1.$$

Proof of Theorem 3. If we use a theorem 3 for gyroquadrilateral $BCNM$ and gyrocollinear points D, A, P, and A (See Figure 5) then the conclusion follows.

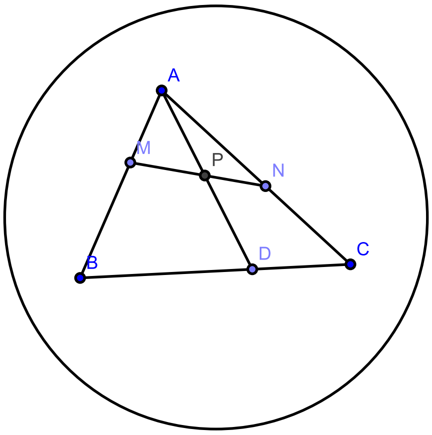

Figure 5

§4. Menelaus's Theorem for n - gons

In this section, we prove Menelaus's theorem for hyperbolic $n-$ gons.

Theorem 5. If l is a gyroline not through any vertex of a $n - gyrogon$ $A_1A_2...A_n$ such that l meets A_1A_2 in M_1, A_2A_3 in $M_2, ...,$ and A_nA_1 in M_n, then

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1. \tag{4}$$

Proof of Theorem 5. We use mathematical induction. For $n = 3$ the theorem is true (see Theorem 2). Let's suppose by induction upon $k \geq 3$ that the theorem is true for any $k - gyrogon$ with $3 \leq k \leq n-1$, and we need to prove it is also true for $k = n$. Suppose a line l intersect the gyroline A_2A_n into the point M. We consider the $n - gyrogon$ $A_1A_2...A_n$ and we split in a $3 - gyrogon$ $A_1A_2A_n$ and $(n-1) - gyrogon$ $A_nA_2A_3...A_{n-1}$ and we can respectively apply the theorem 2 according to our previously hypothesis of induction in each of them, and we respectively get:

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|MA_2|}|MA_2|}{\gamma_{|MA_n|}|MA_n|} \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1,$$

and

$$\frac{\gamma_{|MA_n|}|MA_n|}{\gamma_{|MA_2|}|MA_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_{n-2}A_{n-2}|}|M_{n-2}A_{n-2}|}{\gamma_{|M_{n-2}A_{n-1}|}|M_{n-2}A_{n-1}|} \cdot \frac{\gamma_{|M_{n-1}A_{n-1}|}|M_{n-1}A_{n-1}|}{\gamma_{|M_{n-1}A_n|}|M_{n-1}A_n|} = 1.$$

whence, by multiplying the last two equalities, we get

$$\frac{\gamma_{|M_1A_1|}|M_1A_1|}{\gamma_{|M_1A_2|}|M_1A_2|} \cdot \frac{\gamma_{|M_2A_2|}|M_2A_2|}{\gamma_{|M_2A_3|}|M_2A_3|} \cdot ... \cdot \frac{\gamma_{|M_nA_n|}|M_nA_n|}{\gamma_{|M_nA_1|}|M_nA_1|} = 1.$$

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M class Q composition operators

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Abstract A Hilbert space operator T is of M class $\mathcal{Q}$ if for a fixed real number M$\geq$ 1, T satisfies $\mathrm{M}^2T^{*2}T^2 - 2T^*T + I \geq 0$. M class $\mathcal{Q}$ operators are not necessarily normaloid. We characterize M class $\mathcal{Q}$ Composition operators on L^2 space.

Keywords Class $\mathcal{Q}$ operators, M paranormal Operators, contraction operator, composition operators, conditional expectation.

§1. Introduction and preliminaries

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. An operator $T \in B(H)$ is M Paranormal if for a fixed real number M$>$ 0, T satisfies $\|Tx\|^2 \leq \mathrm{M}\|T^2x\|\|x\|$ for every $x \in H$, normaloid if $r(T) = \|T\|$, where $r(T)$ denotes the spectral radius of T and class $\mathcal{Q}$, $T \in \mathcal{Q}$, if $T^{*2}T^2 - 2T^*T + 1 \geq 0$. Equivalently $T \in \mathcal{Q}$ if $\|Tx\|^2 \leq \frac{1}{2}[\|T^2x\|^2 + \|x\|^2]$ for every $x \in H$. Class $\mathcal{Q}$ operators are inroduced and studied by B. P Duggal et al. [6] and it is well known that every class $\mathcal{Q}$ operator is not necessarily normaloid and every paranormal operator is a normaloid of class $\mathcal{Q}$, that is $P \subseteq \mathcal{Q} \cap N$, where P and N denotes for class of normaloid and paranormal operators. A contraction is an operator T such that $\|T\| \leq 1$ (i.e $\|Tx\| \leq \|x\|$ for every $x \in H$, equivalently $T^*T \leq 1$). By a subspace $\mathcal{M}$ of H, we mean a closed linear manifold of H. A subspace $\mathcal{M}$ is invariant for T if $T(\mathcal{M}) \subseteq \mathcal{M}$ and a part of an operator is a restriction of it to an invariant subspace.

Let (X, Σ, λ) be a sigma-finite measure space, a bounded linear operator $Cf = f \circ T$ on $L^2(X, \Sigma, \lambda)$ is said to be a composition operator induced by T, a non singular measurable transformation from X into it self when the measure λT^{-1} is absolutely continuous with respect to the measure λ and the Radon-Nikodym derivative $d\lambda T^{-1}/d\lambda$=f_0 is essentially bounded. The Radon-Nikodym derivative of the measure $\lambda(T^k)^{-1}$ with respect to λ is denoted by $f_0^{(k)}$, where T^k is obtained by composing T-k times. Every essentially bounded complex valued measurable function f_0 induces the bounded operator M_{f_0} on $L^2(\lambda)$, which is defined by $M_{f_0}f$=f_0f for every $f \in L^2(\lambda)$.

A weighted composition operator(w.c.o)induced by T is defined as Wf=$w(f \circ T)$, w is a complex valued Σ measurable function. Let w_k denote $w(w \circ T)(w \circ T^2) \cdots (w \circ T^{k-1})$ so that W^kf=$w_k(f \circ T)^k$ [11]. To examine the weighted composition operators effectively Alan Lambert [10] associated conditional expectation operator E with T as $E(\cdot/T^{-1}\Sigma)$=$E(\cdot)$. $E(f)$

is defined for each non-negative measurable function $f \in L^p$ $(1 \leqslant p)$ and is uniquely determined by the conditions
(i) $E(f)$ is $T^{-1}\Sigma$ measurable.
(ii) If B is any $T^{-1}\Sigma$ measurable set for which $\int_B f d\lambda$ converges we have $\int_B f d\lambda = \int_B E(f) d\lambda$.
As an operator on L^p, E is the projection on to the closure of range of C. E is the identity on L^p if and only if $T^{-1}\Sigma = \Sigma$. Detailed discussion of E is found in [4, 7, 8].

§2. M class $\mathcal{Q}$ operators

In this section we define operators of M class $\mathcal{Q}$ and consider some basic properties, examples and counterexamples, in order to put this class in its due place. The concept of this class is motivated by Duggal et al. [6]. Recall that for any real λ and any operator $T \in B(H)$,

$$\lambda \mathrm{M}\|T^2x\|\|x\| \leq \frac{1}{2}[\mathrm{M}^2\|T^2x\|^2 + \lambda^2\|x\|^2],$$

and, in particular, for $\lambda = 1$

$$\mathrm{M}\|T^2x\|\|x\| \leq \frac{1}{2}[\mathrm{M}^2\|T^2x\|^2 + \|x\|^2],$$

for every $x \in H$ and fixed real number M≥ 1.

The following alternative definition of M paranormal operator is well known [3, 14]. An operator T is M paranormal if and only if

$$\mathrm{M}^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0,$$

for each $\lambda > 0$. Equivalently T is M paranormal if and only if

$$\lambda\|Tx\|^2 \leq \frac{1}{2}[\mathrm{M}^2\|T^2x\|^2 + \lambda^2\|x\|^2],$$

for every $x \in H$, for all $\lambda > 0$. Note that the above inequalities hold trivially for every $\lambda \leq 0$ for all operator $T \in B(H)$. Take any operator $T \in B(H)$ and for $\mathrm{M} \geq 1$ set

$$\mathcal{Q}_{\mathrm{M}} = \mathrm{M}^2T^{*2}T^2 - 2T^*T + I.$$

Definition 2.1. An operator T is of M class $\mathcal{Q}$ if $\mathcal{Q}_{\mathrm{M}} \geq 0$. Equivalently $T \in$ M *class* $\mathcal{Q}$ if $\|Tx\|^2 \leq \frac{1}{2}[\mathrm{M}^2\|T^2x\|^2 + \|x\|^2]$ for every $x \in H$.

For example, let $x = (x_1, x_2, \cdots) \in l^2$. Define $T : l^2 \to l^2$ by $T(x) = (0, x_1, x_2, \cdots)$, $T^*(x) = (x_2, x_3, \cdots)$. Since $T^{*2}T^2x = (x_1, x_2, \cdots)$ and $T^*Tx = (x_1, x_2, \cdots)$. Then $\mathrm{M}^2T^{*2}T^2 - 2T^*T + 1 \geq 0$, that is $T \in$ M class $\mathcal{Q}$.

Since $\mathrm{M}^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$ if and only if $\lambda^{\frac{1}{2}}T \in$ M class $\mathcal{Q}$ for all $\lambda > 0$.

T is M paranormal if and only if $\lambda T \in$ M class $\mathcal{Q}$ for all $\lambda > 0$.

If M$=1$, then the classes of M paranormal and M class $\mathcal{Q}$ coincides with the class of paranormal and class $\mathcal{Q}$ respectively. The following theorem is immediate from the definition of M class $\mathcal{Q}$.

Theorem 2.2. Let T be a weighted shift with non zero weights $\{\alpha_n\}$ $(n = 0, 1, 2, \cdots)$. T is of M class $\mathcal{Q}$ if and only if $2|\alpha_n|^2 \leq \mathrm{M}^2|\alpha_n|^2|\alpha_{n+1}|^2 + 1$ for each n.

Proof of Theorem 2.2. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of a Hilbert space H. Since $Te_n = \alpha_n e_{n+1}$ and $T^*e_n = \overline{\alpha_{n-1}}e_{n-1}$. An operator T is of M class $\mathcal{Q}$ if and only if $\|Tx\|^2 \leq \frac{1}{2}[\mathrm{M}^2\|T^2x\|^2 + \|x\|^2]$ for every vector $x \in H$ if and only if $\|Te_n\|^2 \leq \frac{1}{2}(\mathrm{M}^2\|T^2e_n\|^2 + \|e_n\|^2)$ for each $n = 1, 2, 3, \cdots$. Since $\|T^2e_n\| = \|T(\alpha_n e_{n+1})\| = |\alpha_n||\alpha_{n+1}|$, we have $T \in$ M class $\mathcal{Q}$ if and only if $2|\alpha_n|^2 \leq \mathrm{M}^2|\alpha_n|^2|\alpha_{n+1}|^2 + 1$.

Corollary 2.3. Let T be a weighted shift with non zero weights $\{\alpha_n\}$ $(n = 0, 1, 2, \cdots)$. T^{-1} is of M class $\mathcal{Q}$ if and only if $\frac{2}{|\alpha_{n-1}|^2} \leq \frac{\mathrm{M}^2}{(|\alpha_{n-1}||\alpha_{n-2}|)^2} + 1$ for each n.

Let H be a separable Hilbert space and let $\langle e_n\rangle_{n=1}^{\infty}$ be an orthonormal basis of H. Define weighted shift on H as $Te_1 = 2e_2$, $Te_2 = 3e_3$, Te_n=e_{n+1}, for all $n \geq 3$. Clearly T is of M class $\mathcal{Q}$ for $\mathrm{M} > 1$ but not class $\mathcal{Q}$. ie., M class $\mathcal{Q}$ (M>1) properly includes class $\mathcal{Q}$.

It is well known that, if T is invertible and hyponormal then T^{-1} is also hyponormal. The same result holds for paranormal and Class $\mathcal{Q}$ operators [6]. However this result does not hold for M Class $\mathcal{Q}$. For example, let $\langle e_n\rangle_{n=1}^{\infty}$ be an orthonormal basis of H. Define weighted shift on H as $Te_1 = 3e_2$, $Te_n = e_{n+1}$, $\forall n \geq 2$. Then clearly $T \in \mathcal{Q}_2$ but $T^{-1} \notin \mathcal{Q}_2$.

Multiple of M class $\mathcal{Q}$ may not be of M class $\mathcal{Q}$. For example $A = \alpha\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in$ M class $\mathcal{Q}$ if $0 < \alpha \leq \frac{1}{\sqrt{2}}$, but for all $\alpha > \frac{1}{\sqrt{2}}$, $A \notin \mathcal{Q}_{\mathrm{M}}$. Clearly $A = \alpha\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is not normaloid for all $\alpha \neq 0$ and hence M class $\mathcal{Q}$ operators are not necessarily Normaloid. Actually $\mathcal{Q}_{\mathrm{M}}$ is not a cone in $B(H)$, although its intersection with the closed unit ball is balanced. Moreover, if T be any M class $\mathcal{Q}$ operator then the tensor product of T and the identity operator I, $T \otimes I$, is of M class $\mathcal{Q}$.

Proposition 2.4. Let $T \in B(H)$ be an operator of M class $\mathcal{Q}$ then the restriction of T to an invariant subspace is again a M class $\mathcal{Q}$ operator.

Proof of Proposition 2.4. Let T be an operator of M class $\mathcal{Q}$ and let $\mathcal{M}$ a T-invariant subspace.

If $y \in \mathcal{M}$, then $2\|(T/\mathcal{M})y\|^2 = 2\|Ty\|^2 \leq \mathrm{M}^2\|T^2y\|^2 + \|y\|^2 = \mathrm{M}^2\|(T/\mathcal{M})^2y\|^2 + \|y\|^2$.

Proposition 2.5. Let T be a Hilbert Space operator.

(i) If $\sqrt{2}T$ is a contraction then $T \in$ M class $\mathcal{Q}$.

(ii) If $T^2 = 0$, then $T \in$ M class $\mathcal{Q}$ if and only if $\|T\| \leq \frac{1}{\sqrt{2}}$.

(iii) If $T \in$ M class $\mathcal{Q}$, $T^2 \neq 0$ and $|\alpha| \leq min\{1, \frac{\|T^2\|^{-1}}{\mathrm{M}}\}$ then $\alpha T \in$ M class $\mathcal{Q}$. In particular $T \in$ M class $\mathcal{Q}$ is a contraction, then $\alpha T \in$ M class $\mathcal{Q}$ whenever $|\alpha| \leq 1$.

Proof of Proposition 2.5.

(i) Since $\sqrt{2}T$ is a contraction, $1 - 2T^*T \geq 0$. Thus $\mathrm{M}^2T^{*2}T^2 - 2T^*T + 1 \geq 0$, that is $T \in$ M class $\mathcal{Q}$.

(ii) If $T^2 = 0$, then $T \in$ M class $\mathcal{Q}$ if and only if $\|T\| \leq \frac{1}{\sqrt{2}}$.

(iii) If $T \in$ M class $\mathcal{Q}$, then $2|\alpha|^2T^*T \leq \mathrm{M}^2|\alpha|^2T^{*2}T^2 + |\alpha|^2I$ for every scalar α, and hence

$$2|\alpha|^2T^*T - |\alpha|^4\mathrm{M}^2T^{*2}T^2 - I \leq (1 - |\alpha|^2)(|\alpha|^2M^2T^{*2}T^2 - I).$$

Suppose $T^2 \neq 0$ and $|\alpha| \leq \frac{\|T^2\|^{-1}}{\mathrm{M}}$ then $\mathrm{M}^2|\alpha|^2T^{*2}T^2 \leq 1$,in addition if $|\alpha| \leq 1$, then $\mathrm{M}^2|\alpha|^4T^{*2}T^2 - 2|\alpha|^2T^*T + I \geq 0$. That is, $\alpha T \in$ M class $\mathcal{Q}$.

Proposition 2.6. A contraction $T \in$ M class $\mathcal{Q}$ is M paranormal if and only if $\mathrm{M}^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0$ for all $\lambda \in (0, 1)$.

Proof of Proposition 2.6. If $T \in$ M class $\mathcal{Q}$ is a contraction, then $\alpha T \in$ M class $\mathcal{Q}$ for all $\alpha \in (0,1]$ or equivalently $\mathrm{M}^2|\alpha|^4 T^{*2}T^2 - 2|\alpha|^2 T^*T + I \geq 0$ and hence, $\mathrm{M}^2 T^{*2}T^2 - \frac{2}{|\alpha|^2}T^*T + \frac{1}{|\alpha|^4}I \geq 0$. Let $\lambda = \frac{1}{|\alpha|^2}$. Then $\lambda \geq 1$ and $0 \leq \mathrm{M}^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I$ for all $\lambda \geq 1$. Thus $T \in$ M class $\mathcal{Q}$ is a contraction, then the above inequality holds for all $\lambda > 0$ if and only if it holds for all $\lambda \in (0,1)$. Thus for a contraction T is of M class $\mathcal{Q}$ is M paranormal if and only if $0 \leq \mathrm{M}^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 I$ for all $\lambda \in (0,1)$.

Every paranormal operator is normaloid of M class $\mathcal{Q}$. That is $P \subseteq Q_{\mathrm{M}} \cap N$. For instance, let $A = \alpha\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, then A is of M class $\mathcal{Q}$ for $0 < \alpha \leq \frac{1}{\sqrt{2}}$, but A is not normaloid (non zero nilpotent) for all $\alpha \neq 0$. Take $T = I \oplus A$, Then clearly T is normaloid but not Paranormal(direct sum of a non zero orthogonal projection and a non zero nilpotent contraction) and hence $T \in \mathcal{Q}_{\mathrm{M}} \cap N \setminus P$ for all $\alpha \in (0, \frac{1}{\sqrt{2}}]$.

§3. M class $\mathcal{Q}$ composition operators

Harrington and Whitely [9] have shown that if $C \in B(L^2(\lambda))$ then $C^*Cf{=}f_0 f$ and $CC^*f{=}(f_0 \circ T)Pf$ for all $f \in L^2$ where P denote the projection of L^2 onto $\overline{R(C)}$. Class $\mathcal{Q}$ composition operators were characterized by S.Panayappan et al. [12]. The following theorem characterize M class $\mathcal{Q}$ composition operators on L^2 space.

Theorem 3.1. Let $C \in B(L^2(\lambda))$. Then C is of M class $\mathcal{Q}$ if and only if $\mathrm{M}^2 f_0^{(2)} - 2f_0 + 1 \geq 0$, a.e.

Proof of Theorem 3.1. Let $C \in B(L^2(\lambda))$ is of M class $\mathcal{Q}$ if and only if $\mathrm{M}^2C^{*2}C^2 - 2C^*C + I \geq 0$. Thus,

$$\langle\ \mathrm{M}^2C^{*2}C^2 - 2C^*C + I)\chi_E, \chi_E\rangle \geq 0$$

for every characteristic function χ_E of E in Σ such that $\lambda(E) < \infty$.

Since $C^{*2}C^2{=}M_{f_0^{(2)}}$ [13] and $C^*C{=}M_{f_0}$ [15], we have $\langle(\mathrm{M}^2 M^2_{f_0^{(2)}} - 2M_{f_0} + 1)\chi_E, \chi_E\rangle \geq 0$.

That is $\int_E (\mathrm{M}^2 f_0^{(2)} - 2f_0 + 1)d\lambda \geq 0$ for every E in Σ.

Hence C is M class $\mathcal{Q}$ if and only if $\mathrm{M}^2 f_0^{(2)} - 2f_0 + 1 \geq 0$, a.e.

For example, let $X{=}\ N$, the set of all natural numbers and λ be the counting measure on it. Define $T : N \to N$ by $T(1){=}T(2){=}1$, $T(3n+m){=}n+1$, $m = 0, 1, 2$ and $n \in N$. Since $\mathrm{M}^2 f_0^{(2)} - 2f_0 + 1 \geq 0$ for every n, C is M class $\mathcal{Q}$ Composition operator.

Corollary 3.2. Let $C \in B(L^2(\lambda))$, Then $C^* \in$ M Class $\mathcal{Q}$ if and only if $\mathrm{M}^2(f_0^{(2)} \circ T^2)P_2 - 2(f_0 \circ T)P_1 + 1 \geq 0$ a.e, where P_1 and P_2 is the projection of L^2 onto $\overline{R(C)}$ and $\overline{R(C^2)}$ respectively.

Proof of Corollary 3.2. We have $C^2C^{*2} = (f_0^{(2)} \circ T^2)P_2$ and $CC^* = (f_0 \circ T)P_1$, so the result follows directly from Theorem 3.1.

Corollary 3.3. Let $C \in B(L^2(\lambda))$ with dense range. Then $C^* \in$ M Class $\mathcal{Q}$ if and only if $\mathrm{M}^2(f_0^{(2)} \circ T^2) - 2f_0 \circ T + 1 \geq 0$, a.e.

Theorem 3.4. If $C \in B(L^2(\lambda))$ is M paranormal then C is of M class $\mathcal{Q}$.

Proof of Theorem 3.4. If C is M paranormal then we see that $\mathrm{M}^2 f_0^{(2)} \geq f_0^2$ a.e [16]. Thus,

$$\mathrm{M}^2 f_0^{(2)} - 2f_0 + 1 \geq f_0^2 - 2f_0 + 1 \geq 0$$

ie., C is of M class $\mathcal{Q}$.

Corollary 3.5. If $C \in B(L^2(\lambda))$ is M quasi hyponormal then C is of M class $\mathcal{Q}$.

Proof of Corollary 3.5. If C is M quasi hyponormal then C is M paranormal by corollary 3.3 [16]. Applying Theorem 3.4, it follows that $C \in \mathcal{Q}_{\mathrm{M}}$.

Now we characterize weighted M class $\mathcal{Q}$ composition operators as follows.

Theorem 3.6. W is M class $\mathcal{Q}$ if and only if $\mathrm{M}^2 f_0^{(2)} E[(w_2^2)] \circ T^{-2} - 2f_0[E(w^2)] \circ T^{-1} + 1 \geq 0$ a.e.

Proof of Theorem 3.6. Since W is of M class $\mathcal{Q}$, $\mathrm{M}^2 W^{*2} W^2 - 2W^*W + 1 \geq 0$ and hence $\langle (\mathrm{M}^2 W^{*2} W^2 - 2W^*W + 1)f, f\rangle \geq 0$ for all $f \in L^2$. Since $W^k f = w_k(f \circ T^k)$ and $W^{*k} f = f_0^{(k)} E(w_k f) \circ T^{-k}$, $W^{*k} W^k = f_0^{(k)} E(w_k^2) \circ T^{-k} f$ and we have $W^*Wf = f_0[E(w^2)] \circ T^{-1} f$ for $w \geq 0$ [4], and hence $\int_E \mathrm{M}^2 (f_0^{(2)} E(w_2^2) \circ T^{-2} - 2f_0[E(w^2)] \circ T^{-1} f + 1) d\lambda \geq 0$ for every $E \in \Sigma$ and so $\mathrm{M}^2 f_0^{(2)} E(w_2^2) \circ T^{-2} - 2f_0[E(w^2)] \circ T^{-1} + 1 \geq 0$ a.e.

Corollary 3.7. Let $T^{-1}\Sigma = \Sigma$. Then W is of M class $\mathcal{Q}$ if and only if $\mathrm{M}^2 f_0^{(2)} (w_2^2) \circ T^{-2} - 2f_0 w^2 \circ T^{-1} + 1 \geq 0$ a.e.

The Aluthge transform of T is the operator $\tilde{T}$ given by $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ was introduced in [1] by Aluthge. More generally we may form the family of operators $\{T_r : 0 < r \leq 1\}$ where $T_r = |T|^r U |T|^{1-r}$ [2]. For a composition operator C, the polar decomposition is given by $C = U|C|$ where $|C|f = \sqrt{f_0} f$ and $Uf = \frac{1}{\sqrt{f_0 \circ T}} f \circ T$. In [5] Lambert has given more generally Aluthge transformation for composition operators as $C_r = |C|^r U |C|^{1-r}$ and $C_r f = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}} f \circ T$. That is C_r is weighted composition operator with weight $\pi = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}}$ where $0 < r < 1$. Since C_r is a weighted composition operator it is easy to show that $|C_r|f = \sqrt{f_0[E(\pi)^2 \circ T^{-1}]} f$ and $|C_r^*| f = vE[vf]$ where $v = \frac{\pi\sqrt{f_0 \circ T}}{[E(\pi\sqrt{f_0 \circ T})^2]^{\frac{1}{4}}}$. Also we have

$C_r^k f = \pi_k (f \circ T^k)$,
$C_r^{*k} f = f_0^{(k)} E(\pi_k f) \circ T^{-k}$,
$C_r^{*k} C_r^k f = f_0^{(k)} E(\pi_k^2) \circ T^{-k} f$.

Corollary 3.8. Let $C_r \in B(L^2(\lambda))$. Then C_r is of M class $\mathcal{Q}$ if and only if $\mathrm{M}^2 f_0^{(2)} E(\pi_2^2) \circ T^{-2} - 2f_0 E(\pi^2) \circ T^{-1} + 1 \geq 0$ a.e.

Proof of Corollary 3.8. Since C_r is weighted composition operator with weight $\pi = (\frac{f_0}{f_0 \circ T})^{\frac{r}{2}}$, it follows that C_r is of M class $\mathcal{Q}$ if and only if $M^2 f_0^{(2)} E(\pi_2^2) \circ T^{-2} - 2f_0 E(\pi^2) \circ T^{-1} + 1 \geq 0$ a.e.

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An inequality on density matrix and its a generalization [1]

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Abstract In this note, we introduce a generalized trace from a $*$-ideal I of a unital C^*-algebra A into a unital Abelian C^*-algebra B and prove some properties on it. Our main result extends a Grüss-type inequality on density matrices to the setting of C^*-algebras.

Keywords Density matrix, inequality, C^*-algebra, generalized trace.

§1. Introduction

In a quantum system, the states are represented by density matrices, which are positive semi-definite matrices of trace 1 [1−4]. Recently, P. F. Renaud in [5] proved a matrix formulation of Grüss inequality which says that if A and B are $n \times n$ complex matrices and the numerical ranges $W(A)$ and $W(B)$ are contained in disks of radii R and S, respectively, then for every $n \times n$ density matrix T, the following inequality holds

$$|\mathrm{Tr}(TAB) - \mathrm{Tr}(TA)\mathrm{Tr}(TB)| \leq 4RS. \tag{1.1}$$

In this note, we introduce a generalized trace from a $*$-ideal $I$of a unital C^*-algebra A into a unital Abelian C^*-algebra B and prove some properties on it. Our main results extends the Grüss-type inequality (1.1).

Throughout this note, we assume that A is a unital C^*-algebra with unit 1_A and I is a *-ideal of A, I^+ is the set of all positive elements in I and $U(A)$ is the group of unitary elements of A. $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{C}$ is the complex field. B is an unital Abelian C^*-algebra with unit 1_B and $\Omega(B)$ is the character space of B that is the space of all characters of B. For every element b in B, put $|b| = (b^*b)^{\frac{1}{2}}$, called the absolute value of b. Let $U : A \longrightarrow \mathrm{B}(H)$ be the universal representation. For a in A, define the numerical range $W(a)$ of a to be that of operator $U(a)$, that is,

$$W(a) = \{\langle U(a)x, x\rangle : x \in H, \|x\| = 1\}.$$

From [7], we have

$$\frac{1}{2}\|a\| \leq \omega(a) \leq \|a\|, \forall a \in A, \tag{1.2}$$

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where $\omega(a) := \sup\{|\langle U(a)x, x\rangle| : x \in H, \|x\| = 1\}$, called the numerical radius of a. It is known that if a is normal (i.e. $aa^* = a^*a$), then $\omega(a) = \|a\|$.

Definition 1.1. A linear mapping $\tau : I \longrightarrow B$ is called a generalized trace if it satisfies the following conditions:

(T1) τ is positive, i.e., $a \in I^+ \Longrightarrow \tau(a) \geq 0$;

(T2) τ is unitary-stable, i.e., $\forall u \in U(A)$ and $\forall a \in I$,

$$\tau(u^*au) = \tau(a);$$

(T3) τ is $*-$preserving, i.e., $\forall a \in I, \tau(a^*) = (\tau(a))^*$;

(T4) τ is sub-Jordan, i.e., $\forall a \in I, \tau(a^2) \leq (\tau(a))^2$.

Clearly, if H is a Hilbert space over $\mathbb{C}$, $A = B(H)$ (the C^*-algebra of all bounded linear operators on H), $I = T(H)$ (the $*$-deal of all trace-class operators on H) or $A = I = M_n(\mathbb{C})$, then the usual trace $\mathrm{Tr} : I \longrightarrow \mathbb{C}$ is a generalized trace. To give a non-trivial example of generalized trace, let us consider such a C^*- algebra A which has proper commutator ideal $k(A)$ that is the closed ideal of Agenerated by all commutators $ab - ba$ for all a and b in A. It is known that the Toeplitz algebra is such an algebra [6, pp. 102]. In this case, the quotient algebra $A/k(A)$ is a nonzero unital Abelian C^*-algebra. Let $\pi : A \rightarrow A/k(A)$ be the quotient homomorphism and ϕ be any character on $A/k(A)$. Then for every x in $A/k(A)$ with $x \geq 1$, the mapping

$$\tau : A \longrightarrow B, \tau(a) := \phi(x \cdot \pi(a))1_B$$

is a generalized trace. Moreover, if x is the unit of $A/k(A)$ and t is a trace element of A (i.e., $t \geq 0$ and $\tau(t) = 1_B$), then

$$0 \leq \tau(ta) \leq \|a\|1_B, \forall a \geq 0. \tag{1.3}$$

Another example is the matrix-trace tr from C^*-algebra $M_n(A)$into A, see [8] for the details.

§2. Main results

In the sequel, let us assume that $\tau : I \longrightarrow B$ is a generalized trace. With these notations, we have the following main results.

Proposition 2.1.

(a) $\forall a \in I, \forall b \in A$ and $n \in \mathbb{N}$, we have $\tau((ab)^n) = \tau((ba)^n)$,

(b) $\tau((ab)^n) \geq 0,\ \forall a, b \in I^+, n \in \mathbb{N}$,

(c) $|\tau(b^*a)|^2 \leq \tau(a^*a)\tau(b^*b),\ \forall a, b \in I$,

(d) $0 \leq \tau(ab) \leq \tau(a)\tau(b),\ \forall a, b \in I^+$,

(e) $0 \leq \tau(a^n) \leq (\tau(a))^n,\ \forall a \in I^+, n \in \mathbb{N}$.

Proof. (a) $\forall u \in U(A)$ and $\forall a \in I$, from the property (T2) we have

$$\tau(ua) = \tau(u^*(ua)u) = \tau(au).$$

It follows from this and the property A =span($U(A)$) that the statement (a) is true for $n = 1$. Thus, $\forall a \in I, \forall b \in A$ and $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\tau((ab)^n) &= \tau((a \cdot b(ab)^{n-1}) \\ &= \tau(b(ab)^{n-1} \cdot a) \\ &= \tau((ba)^n).\end{aligned}$$

(b) It is easy to see that $\forall a, b \in I^+$ and $\forall n \in \mathbb{N}, (ab)^{n-1}a$ is in I^+. Thus, by (a) and (T1) we have

$$\tau((ab)^n) = \tau(b^{\frac{1}{2}} \cdot (ab)^{n-1}a \cdot b^{\frac{1}{2}}) \geq 0.$$

(c) Let ϕ be an arbitrary element of $\Omega(B)$ and define $\langle a, b\rangle = \phi(\tau(b^*a))$. Then we obtain asemi-inner product on I. Thus, $\forall a, b \in I$, we have $|\langle a, b\rangle|^2 \leq \langle a, a\rangle\langle b, b\rangle$, i.e.,

$$|\phi(\tau(b^*a))|^2 \leq \phi(\tau(a^*a))\phi(\tau(b^*b)),$$

thus,

$$\phi(|\tau(b^*a)|^2) \leq \phi(\tau(a^*a)\tau(b^*b)).$$

This shows the desired inequality.

(d) $\forall a, b \in I^+$, we see from (b), (c) and (T4) that

$$0 \leq \tau(ab) \leq [\tau(a^*a)\tau(b^*b)]^{\frac{1}{2}} \leq \tau(a)\tau(b).$$

(e) It is from (d). The proof is then completed.

Theorem 2.2. Let t be a trace element of I (i.e., $t \geq 0$ and $\tau(t) = 1_B$) satisfying (1.3), then

(a) For all a and b in A,

$$|\tau(tab^*)|^2 \leq \tau(taa^*)\tau(tbb^*). \tag{2.1}$$

(b) For all $a, b \in A$, we have

$$|\tau(tab) - \tau(ta)\tau(tb)| \leq \mathrm{dist}(a, \mathbb{C}1_A) \cdot \mathrm{dist}(b, \mathbb{C}1_A) \cdot 1_B \leq \|a\|\|b\|1_B. \tag{2.2}$$

(c) If $a, b \in A$ such that $W(a), W(b)$ are contained in the disks of radii r, s, respectively, then we have

$$|\tau(tab) - \tau(ta)\tau(tb)| \leq 4rs \cdot 1_B. \tag{2.3}$$

In addition, if a, b are normal, then

$$|\tau(tab) - \tau(ta)\tau(tb)| \leq rs \cdot 1_B. \tag{2.4}$$

Proof. Note that for any x and y in B, $x \leq y$ if and only if $\phi(x) \leq \phi(y)(\forall \phi \in \Omega(B)$. Let $\phi \in \Omega(B)$, define $\langle\cdot,\cdot\rangle : A \times A \longrightarrow \mathbb{C}$ as

$$\langle x, y\rangle := \phi(\tau(txy^*)). \tag{2.5}$$

Thus, we have the following Cauchy-Schwarz inequality: $\forall x, y \in A$,

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle. \tag{2.6}$$

Especially, we have $\forall x \in A$,

$$|\langle x, 1_A\rangle|^2 \leq \langle x, x\rangle\langle 1_A, 1_A\rangle = \langle x, x\rangle. \tag{2.7}$$

(a) $\forall \phi \in \Omega(B)$, from (2.6) we obtain

$$\begin{aligned}\phi(|\tau(tab^*)|^2) &= |\langle a, b\rangle|^2 \\ &\leq \langle a, a\rangle\langle b, b\rangle \\ &= \phi(\tau(taa^*))\phi(\tau(tbb^*)) \\ &= \phi(\tau(taa^*)\tau(tbb^*)).\end{aligned}$$

Hence, (2.1) has been proved.

(b) Fixed a $\phi \in \Omega(B)$. It suffices to prove that $\forall \lambda, \mu \in \mathbb{C}$,

$$\phi(|\tau(tab)) - \tau(ta)\tau(tb)|) \leq \|a - \lambda\|\|b - \mu\|. \tag{2.8}$$

Let $\lambda, \mu \in \mathbb{C}$ and define $a_\lambda = a - \lambda, b_\mu = b - \mu$. Then from (2.6) and (2.7) we have

$$\begin{aligned}&\phi(|\tau(tab) - \tau(ta))\tau(tb)|^2) \\ =\ & |\phi(\tau(tab)) - \phi(\tau(ta))\phi(\tau(tb))|^2 \\ =\ & |\langle a, b^*\rangle - \langle a, 1_A\rangle\langle b, 1_A\rangle|^2 \\ =\ & |\langle a - \langle a, 1_A\rangle 1_A, b^* - \langle b^*, 1_A\rangle 1_A\rangle|^2 \\ \leq\ & \langle a - \langle a, 1_A\rangle 1_A, a - \langle a, 1_A\rangle 1_A\rangle \cdot \langle b^* - \langle b^*, 1_A\rangle 1_A, b^* - \langle b^* 1_A\rangle 1_A\rangle \\ =\ & [\langle a, a\rangle - |\langle a, 1_A\rangle|^2] \cdot [\langle b^*, b^*\rangle - |\langle b^*, 1_A\rangle|^2] \\ =\ & [\langle a_\lambda, a_\lambda\rangle - |\langle a_\lambda, 1_A\rangle|^2] \cdot [\langle b_\mu^*, b_\mu^*\rangle - |\langle b_\mu^*, 1_A\rangle|^2] \\ \leq\ & \langle a_\lambda, a_\lambda\rangle \cdot \langle b_\mu^*, b_\mu^*\rangle \\ =\ & \phi(\tau(ta_\lambda a_\lambda^*))\phi(\tau(tb_\mu^* b_\mu)) \\ \leq\ & \phi(\|a_\lambda\|^2 1_B)\phi(\|b_\mu\|^2 1_B) \\ =\ & (\|a_\lambda\|\|b_\mu\|)^2.\end{aligned}$$

This shows that (2.8) holds.

(c) Let $a, b \in A$ such that $W(a)$, $W(b)$ are contained in closed disks $D(\lambda_0, r)$, $D(\mu_0, s)$ with radii r, s and centered λ_0, μ_0, respectively. Hence,

$$W(a_{\lambda_0}) \subset D(\lambda_0, r) - \lambda_0 = D(0, r),$$

$$W(b_{\mu_0}) \subset D(\mu_0, s) - \mu_0 = D(0, s).$$

Thus, from (1.2) we have

$$\|a - \lambda_0\|\|b - \mu_0\| \leq 4\omega(a_{\lambda_0})\omega(b_{\mu_0}) \leq 4rs. \tag{2.9}$$

From (2.2) and (2.9), we obtain (2.3). In the case where a, b are normal, then a_{λ_0}, b_{μ_0} are normal. Thus $\|a - \lambda_0\|\|b - \mu_0\| = \omega(a_{\lambda_0})\omega(b_{\mu_0}) = rs$. Therefore (2.2) yields (2.4). This completes the proof.

Remark 2.1. For a subset E of A and a character$\phi \in \Omega(B)$, define

$$\delta(E, t, \phi) = \sup_{a,b \in E \setminus \{0\}} \frac{\phi(|\tau(tab) - \tau(ta)\tau(tb)|)}{\omega(a)\omega(b)}.$$

Then by Theorem 2.2 (c), we get $\delta(A, t, \phi) \leq 4$. If $\mathrm{Nor}(A)$ is a set of all normal elements of A, then from Theorem 2.2 (c), we see that $\delta(\mathrm{Nor}(A), t, \phi) \leq 1$.

Remark 2.2. For each $\phi \in \Omega(B)$, define $F_t(a, b) = \tau(tab) - \tau(ta)\tau(tb)$, then we obtain a bilinear mapping $F_t : A \times A \to B$. Using (2.2), we see that F_t is continuous and satisfies $\|F_t\| \leq 1$.

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A proof of Smarandache-Patrascu's theorem using barycentric coordinates

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Abstract In this article we prove the Smarandache-Patrascu's theorem in relation to the inscribed orthohomological triangles using the barycentric coordinates.

Keywords Smarandache-Patrascu's theorem, barycentric coordinates.

Definition. Two triangles and ABC and $A_1B_1C_1$, where $A_1 \in BC$, $B_1 \in AC$, $C_1 \in AB$, are called inscribed ortho homological triangles if the perpendiculars in A_1, B_1, C_1 on BC, AC, AB respectively are concurrent.

Observation. The concurrency point of the perpendiculars on the triangle ABC's sides from above definition is the orthological center of triangles ABC and $A_1B_1C_1$.

Smarandache-Patrascu Theorem. If the triangles ABC and $A_1B_1C_1$ are orthohomological, then the pedal triangle $A_1^{'}B_1^{'}C_1^{'}$ of the second center of orthology of triangles ABC and $A_1B_1C_1$, and the triangle ABC are orthohomological triangles.

Proof. Let $P(\alpha, \beta, \gamma)$, $\alpha + \beta + \gamma = 1$, be the first orthologic center of triangles ABC and $A_1B_1C_1$ (See Figure 1).

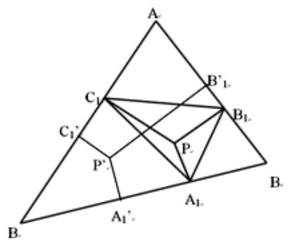

Fig. 1

The perpendicular vectors on the sides are:

$$U_{BC}^{\perp} = \left(2a^2, -a^2 - b^2 + c^2, -a^2 + b^2 - c^2\right),$$

$$U_{CA}^{\perp} = \left(-a^2 - b^2 + c^2, 2b^2, a^2 - b^2 - c^2\right),$$

$$U_{AB}^{\perp} = \left(-a^2 + b^2 - c^2, a^2 - b^2 - c^2, 2c^2\right).$$

We know that:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} = -1, \tag{1}$$

and we want to prove that:

$$\frac{\overrightarrow{A_1'B}}{\overrightarrow{A_1'C}} \cdot \frac{\overrightarrow{B_1'C}}{\overrightarrow{B_1'A}} \cdot \frac{\overrightarrow{C_1'A}}{\overrightarrow{C_1'B}} = -1. \tag{2}$$

We will show that:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} \cdot \frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} \cdot \frac{\overrightarrow{A_1'B}}{\overrightarrow{A_1'C}} \cdot \frac{\overrightarrow{B_1'C}}{\overrightarrow{B_1'A}} \cdot \frac{\overrightarrow{C_1'A}}{\overrightarrow{C_1'B}} = 1.$$

implies the relation (2)

The equation of the line BC is x=0, and the equation of the line PA_1 is

$$\begin{vmatrix} 0 & y & z \\ \alpha & \beta & \gamma \\ 2a^2 & -a^2-b^2+c^2 & -a^2+b^2-c^2 \end{vmatrix} = 0.$$

It results that:

$$y \cdot \begin{vmatrix} \alpha & \gamma \\ 2a^2 & -a^2+b^2+c^2 \end{vmatrix} = z \cdot \begin{vmatrix} \alpha & \beta \\ 2a^2 & -a^2-b^2+c^2 \end{vmatrix} = 0.$$

Because y+z=1, we find:

$$A_1\left(0, \frac{\alpha}{2a^2}(a^2+b^2-c^2)+\beta, \frac{\alpha}{2a^2}(a^2-b^2-c^2)+\gamma\right).$$

Similarly:

$$B_1\left(\frac{-\beta}{2b^2}(-a^2-b^2+c^2)+\alpha, 0, \frac{-\beta}{2b^2}(a^2-b^2-c^2)+\gamma\right),$$

$$C_1\left(\frac{-\gamma}{2c^2}(-a^2+b^2-c^2)+\alpha, \frac{\gamma}{2c^2}(a^2-b^2-c^2)+\beta, 0\right).$$

We will make the following notations:

$$-a^2+b^2-c^2 = i, -a^2-b^2+c^2 = j, a^2-b^2-c^2 = k,$$

And we have:

$$A_1\left(0, \frac{-\alpha}{2a^2}j+\beta, \frac{-\alpha}{2a^2}i+\gamma\right),$$

$$B_1\left(\frac{-\beta}{2b^2}j+\alpha, 0, \frac{-\beta}{2b^2}k+\gamma\right),$$

$$C_1\left(\frac{-\gamma}{2c^2}i+\alpha, \frac{-\gamma}{2c^2}k+\beta, 0\right),$$

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}} = -\frac{\frac{-\alpha}{2a^2}i+\gamma}{\frac{-\alpha}{2a^2}j+\beta};$$

$$\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}} = -\frac{\frac{-\beta}{2b^2}j+\alpha}{\frac{-\beta}{2b^2}k+\gamma};$$

$$\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}} = -\frac{\frac{-\gamma}{2c^2}k+\beta}{\frac{-\gamma}{2c^2}i+\alpha}.$$

If $P^{'}\left(\alpha^{'},\beta^{'},\gamma^{'}\right)$ is the second center of orthology of the triangles ABC and $A_1B_1C_1$, and $A_1^{'},B_1^{'},C_1^{'}$ are the projections of $P^{'}$ on BC, AC, AB respectively, similarly, we will find:

$$\frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}} = -\frac{\frac{-\alpha^{'}}{2a^2}i+\gamma^{'}}{\frac{-\alpha^{'}}{2a^2}j+\beta^{'}};$$

$$\frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}} = -\frac{\frac{-\beta^{'}}{2b^2}j+\alpha^{'}}{\frac{-\beta^{'}}{2b^2}k+\gamma^{'}};$$

$$\frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}} = -\frac{\frac{-\gamma^{'}}{2c^2}k+\beta^{'}}{\frac{-\gamma^{'}}{2c^2}i+\alpha^{'}}.$$

It is known the theorem [2].

Theorem. Given two isogonal conjugated points $P(\alpha,\beta,\gamma)$ and $P^{'}(\alpha^{'},\beta^{'},\gamma^{'})$ with respect to the triangle ABC (BC=a, CA=b, AB=c), then:

$$\frac{\alpha\alpha^{'}}{a^2}=\frac{\beta\beta^{'}}{b^2}=\frac{\gamma\gamma^{'}}{c^2}.$$

On the other side:

$$\frac{\overrightarrow{A_1B}}{\overrightarrow{A_1C}}\cdot\frac{\overrightarrow{A_1^{'}B}}{\overrightarrow{A_1^{'}C}} = \frac{\left(\frac{-\alpha}{2a^2}i+\gamma\right)\left(\frac{-\alpha^{'}}{2a^2}i+\gamma^{'}\right)}{\left(\frac{-\alpha}{2a^2}j+\beta\right)\left(\frac{-\alpha^{'}}{2a^2}j+\beta^{'}\right)} = \frac{\frac{\alpha\alpha^{'}}{4a^4}i^2-\frac{\alpha\gamma^{'}}{2a^2}i-\frac{\alpha^{'}\gamma}{2a^2}i+\gamma\gamma^{'}}{\frac{\alpha\alpha^{'}}{4a^4}j^2-\frac{\alpha\beta^{'}}{2a^2}j-\frac{\alpha^{'}\beta}{2a^2}j+\beta\beta^{'}} = \frac{U_1}{V_1};$$

$$\frac{\overrightarrow{B_1C}}{\overrightarrow{B_1A}}\cdot\frac{\overrightarrow{B_1^{'}C}}{\overrightarrow{B_1^{'}A}} = \frac{\frac{\beta\beta^{'}}{4b^4}j^2-\frac{\beta\alpha^{'}}{2b^2}j-\frac{\alpha\beta^{'}}{2b^2}j+\alpha\alpha^{'}}{\frac{\beta\beta^{'}}{4b^4}k^2-\frac{\beta\gamma^{'}}{2b^2}k-\frac{\beta^{'}\gamma}{2b^2}k+\gamma\gamma^{'}} = \frac{U_2}{V_2};$$

$$\frac{\overrightarrow{C_1A}}{\overrightarrow{C_1B}}\cdot\frac{\overrightarrow{C_1^{'}A}}{\overrightarrow{C_1^{'}B}} = \frac{\frac{\gamma\gamma^{'}}{4c^4}k^2-\frac{\gamma\beta^{'}}{2c^2}k-\frac{\gamma^{'}\beta}{2c^2}k+\beta\beta^{'}}{\frac{\gamma\gamma^{'}}{4c^4}i^2-\frac{\gamma\alpha^{'}}{2c^2}i-\frac{\gamma^{'}\alpha}{2c^2}i+\alpha\alpha^{'}} = \frac{U_3}{V_3}.$$

The only thing left to be proved is that:

$$\frac{U_1}{V_1}\cdot\frac{U_2}{V_2}\cdot\frac{U_3}{V_3}=1$$

if and only if

$$\frac{\frac{a^2}{c^2}U_1}{V_1}\cdot\frac{\frac{b^2}{a^2}U_2}{V_2}\cdot\frac{\frac{c^2}{b^2}U_3}{V_3}=1.$$

We show that

$$\frac{b^2}{a^2}U_2=V_1,\ \frac{c^2}{b^2}U_3=V_2,\ \frac{a^2}{c^2}U_1=V_3;$$

$$\frac{b^2}{a^2}U_2=\frac{\beta\beta^{'}}{4a^2b^2}j^2-\frac{\beta\alpha^{'}}{2a^2}j-\frac{\alpha\beta^{'}}{2a^2}j+\frac{b^2}{a^2}\alpha\alpha^{'}=\frac{\alpha\alpha^{'}}{4a^4}j^2-\frac{\beta\alpha^{'}}{2a^2}j-\frac{\beta^{'}\alpha}{2a^2}j+\beta\beta^{'}=V_1;$$

$$\frac{c^2}{b^2}U_3 = \frac{\gamma\gamma^{'}}{4c^2b^2}k^2 - \frac{\gamma\beta^{'}}{2b^2}k - \frac{\gamma^{'}\beta}{2b^2}k + \frac{c^2}{b^2}\beta\beta^{'} = \frac{\beta\beta^{'}}{4b^4}k^2 - \frac{\gamma\beta^{'}}{2b^2}k - -\frac{\gamma^{'}\beta}{2b^2}k + \gamma\gamma^{'} = V_2;$$

$$\frac{a^2}{c^2}U_1 = \frac{\alpha\alpha^{'}}{4a^2c^2}i^2 - \frac{\alpha\gamma^{'}}{2c^2}i - \frac{\alpha^{'}\gamma}{2c^2}i + \frac{a^2}{c^2}\gamma\gamma^{'} = \frac{\gamma\gamma^{'}}{4c^4}i^2 - \frac{\alpha\gamma^{'}}{2c^2}i - -\frac{\alpha^{'}\gamma}{2c^2}i + \alpha\alpha^{'} = V_3.$$

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Inequalities and monotonicity for the ration of k-gamma functions

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Abstract Let $x > 0, y \geq 0$ be real numbers. The function $f(x) = \frac{[\Gamma_k(x+y+1)/\Gamma_k(y+1)]^{\frac{1}{x}}}{x+y+1}$ is strictly decreasing and strictly logarithmically convex on $(0, \infty)$. Moreover $\lim_{x \to 0} f(x) = \frac{e^{\psi_k(y+1)}}{y+1}$ and $\frac{x+y+1}{x+y+2} \leq \frac{\left[\Gamma_k(x+y+1)/\Gamma_k(y+1)\right]^{\frac{1}{x}}}{\left[\Gamma_k(x+y+2)/\Gamma_k(y+1)\right]^{\frac{1}{x+1}}}$.

Keywords Γ_k function, inequalities.

§1. Introduction and preliminaries

The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The digamma (or psi) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [11]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} + \sum_{n \geq 1} \frac{x}{n(n+x)}. \tag{1}$$

Rafael Diaz and Eddy Pariguan (see [12]) defined function Γ_k admits an infinite product expression given by

$$\frac{1}{\Gamma_k(x)} = x k^{-\frac{x}{k}} e^{\frac{x}{k}\gamma} \prod_{n=1}^\infty \left(\left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}} \right), \tag{2}$$

where $\Gamma_k \to \Gamma$ as $k \to 1$.

We define the k-analogue of the psi function as the logarithmic derivative of the Γ_k function, that is

$$\psi_k(x) = \frac{d}{dx} \ln \Gamma_k(x) = \frac{\Gamma_k'(x)}{\Gamma_k(x)}, \quad k > 0. \tag{3}$$

In [1], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{\frac{1}{r}}$, then

$$1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r}, \tag{4}$$

which can be rearranged as

$$[\Gamma(1+r)]^{\frac{1}{r}} < [\Gamma(2+r)]^{\frac{1}{r+1}}, \tag{5}$$

and

$$\frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r} > \frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}. \tag{6}$$

In [2, 3], H. Alzer and J. S. Martins refined the right inequality in (4) and showed that, if n is a positive integer, then for all positive real numbers r, we have

$$\frac{n}{n+1} < \left(\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r} \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{7}$$

Both bounds in (7) are the best possible.

The inequalities in (4) were refined and generalized in [4, 5, 7, 9] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{\frac{1}{n}} \Big/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{\frac{1}{(n+m)}} \leq \sqrt{\frac{n+k}{n+m+k}}, \tag{8}$$

where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (8) is valid.

In [4], inequalities in (8) were generalized and Feng Qi obtained the following inequalities on the ration for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer k and natural numbers n and m:

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left(\prod_{i=k+1}^{n+k}(i+\alpha)\right)^{\frac{1}{n}}}{\left(\prod_{i=k+1}^{n+m+k}(i+\alpha)\right)^{\frac{1}{(n+m)}}} \leq \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}}, \tag{9}$$

where $\alpha \in [0,1]$ is a constant. For $n = m = 1$, the equality in (6) is valid.

Furthermore, for nonnegative integer k and natural numbers n and m, we have

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left(\prod_{i=k+1}^{n+k}(ai+b)\right)^{\frac{1}{n}}}{\left(\prod_{i=k+1}^{n+m+k}(ai+b)\right)^{\frac{1}{(n+m)}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \tag{10}$$

where a is a positive constant and b is a nonnegative integer. For $n = m = 1$, the equality in (10) is valid, (see [6]).

It is clear that inequalities in (10) extend those in (9).

In [10], the following monotonicity results for the Gamma function were established. The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x > 0$, which

recovers the inequalities in (4) which refers to integer value of r. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})]^x$ decreases for $0 < x < 1$, where $\gamma = 0.57721566\cdots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [5], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{11}$$

is decreasing for $x \geq 1$, for fixed $y \geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}}. \tag{12}$$

§2. Main results

The following Theorem is the main result of these notes.

Lemma 2.1. a) The function $\psi_k(x)$ defined by (3) has the following series representation

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(x+nk)}. \tag{13}$$

b) The function $\psi_k^{'}$ is strictly completely monotonic on $(0,\infty)$.

Proof. After logarithmical and derivative (2) we take (13).

b) Deriving n times the relation (13) one finds that

$$\psi_k^{(n)} = (-1)^{n+1} \cdot n! \sum_{p=0}^{\infty} \frac{1}{(x+pk)^{n+1}}, \tag{14}$$

hence $(-1)^n(\psi_k^{'}(x))^{(n)} > 0$, for $x > 0$ and $n \geq 0$.

Remark 2.2. We note that $\lim\limits_{k\to 1} \psi_k^{(n)} = \psi^{(n)}(x)$.

Theorem 2.3. Let $x > 0, y \geq 0$ be real numbers,and $k \geq 1$ then the function

$$f(x) = \frac{[\Gamma_k(x+y+1)/\Gamma_k(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{15}$$

is strictly decreasing on $(0,\infty)$. Moreover

$$\lim_{x\to 0} f(x) = \frac{e^{\psi_k(y+1)}}{y+1},$$

and

$$\frac{x+y+1}{x++y+2} \leq \frac{\Big[\Gamma_k(x+y+1)/\Gamma_k(y+1)\Big]^{\frac{1}{x}}}{\Big[\Gamma_k(x+y+2)/\Gamma_k(y+1)\Big]^{\frac{1}{x+1}}}.$$

Proof. Taking logarithm yields

$$\ln f(x) = \frac{1}{x}\Big[\ln\Gamma_k(x+y+1) - \ln\Gamma_k(y+1)\Big] - \ln(x+y+1).$$

For $x > 0$, define

$$h(x) = x^2 \frac{f'(x)}{f(x)} = -\ln \frac{\Gamma_k(x+y+1)}{\Gamma_k(y+1)} + x\psi_k(x+y+1) - \frac{x^2}{x+y+1}.$$

Differentiation of h gives.

$$\begin{aligned}
\frac{1}{x}h'(x) &= \psi_k'(x+y+1) - \frac{1}{x+y+1} - \frac{y+1}{(x+y+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{(x+y+nk)^2} - \frac{1}{x+y+1} - \frac{y+1}{(x+y+1)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{(x+y+nk)^2} - \sum_{n=1}^{\infty} \Big[\frac{1}{x+y+n} - \frac{1}{(x+y+n+1)}\Big] - \\
&- \sum_{n=1}^{\infty} \Big[\frac{y+1}{(x+n+y)^2} - \frac{y+1}{(x+y+n+1)^2}\Big] \\
&< \int_0^{\infty} \frac{t}{1-e^{-t}} e^{-(x+y+1)t} dt - \sum_{n=1}^{\infty} \Big[\frac{1}{x+y+n} - \frac{1}{(x+y+n+1)}\Big] - \\
&- \sum_{n=1}^{\infty} \Big[\frac{y+1}{(x+n+y)^2} - \frac{y+1}{(x+y+n+1)^2}\Big] \\
&= -\sum_{n=1}^{\infty} \Big[\frac{y}{(x+y+n)^2} + \frac{1}{(x+y+n)(x+y+n+1)} - \frac{y+1}{(x+y+n+1)^2}\Big] \\
&= -\sum_{n=1}^{\infty} \frac{(2y+1)(x+y+n)+y}{(x+y+n)^2(x+y+n+1)^2} < 0.
\end{aligned}$$

Hence, the function h is strictly decreasing and $h(x) < h(0) = 0$, for $x > 0$, which yields the desired results that $f'(x) < 0$.

$$\ln f(x) = \frac{1}{x}\Big[\ln \Gamma_k(x+y+1) - \ln \Gamma_k(y+1)] - \ln(x+y+1). \tag{16}$$

By L. Hospital rule, we conclude that

$$\lim_{x\to 0} f(x) = \frac{e^{\psi_k(y+1)}}{y+1}, y \geq 0. \tag{17}$$

Theorem 2.4. The function f given by (15) is strictly logarithmically convex on $(0, \infty)$ for $k \geq 1$.

Proof. Define for $x > 0$

$$\begin{aligned}
g(x) &= x^3 \frac{d^2[\ln f(x)]}{dx^2} \\
&= 2\ln \frac{\Gamma_k(x+y+1)}{\Gamma_k(y+1)} - 2x\psi_k(x+y+1) + +x^2\psi_k'(x+y+1) + \frac{x^3}{(x+y+1)^2}.
\end{aligned}$$

Differentiation of g yields

$$\begin{aligned}
\frac{1}{x^2}g^{'}(x) &= \psi^{''}(x+y+1)+\frac{1}{(x+y+1)^2}+\frac{2(y+1)}{(x+y+1)^3}\\
&= -\sum_{n=1}^{\infty}\frac{2}{(x+s+nk)^3}+\sum_{n=1}^{\infty}\Big[\frac{1}{(x+y+n)^2}-\frac{1}{(x+y+n+1)^2}\Big]\\
&+\sum_{n=1}^{\infty}\Big[\frac{2(y+1)}{(x+y+n)^2}-\frac{2(y+1)}{(x+y+n+1)^2}\Big]\\
&> \int_0^{\infty}\frac{t^2}{e^{-t}-1}e^{-(x+y+1)t}dt+\sum_{n=1}^{\infty}\Big[\frac{1}{(x+y+n)^2}-\frac{1}{(x+y+n+1)^2}\Big]\\
&+\sum_{n=1}^{\infty}\Big[\frac{2(y+1)}{(x+y+n)^2}-\frac{2(y+1)}{(x+y+n+1)^2}\Big].\\
&= \sum_{n=1}^{\infty}\frac{3(2y+1)(x+y+n)^2+(6y+1)(x+y+n)+2y}{(x+y+n)^3(x+y+n+1)^3}>0.
\end{aligned}$$

Hence the function g is strictly increasing and $g(x) > g(0) = 0$, for $x > 0$, which yields the desired results, that is $\dfrac{d^2(\ln f(x))}{dx^2} > 0$ for $x > 0$.

Corollary 1. Let $y \geq 0$ be a real number. Then for all real numbers $x > 0$

$$\frac{\Big[\Gamma_k(x+y+1)\Big/\Gamma_k(y+1)\Big]^{\frac{1}{x}}}{x+y+1}\leq\frac{e^{\psi_k(y+1)}}{y+1}. \tag{18}$$

Proof. Since f is decreasing and

$$\lim_{x\to 0}f(x)=\frac{e^{\psi_k(y+1)}}{y+1}, \tag{19}$$

we obtain

$$f(x)\leq\lim_{x\to 0}f(x)=\frac{e^{\psi_k(y+1)}}{y+1}$$

and proof is complete.

§3. Open problem

At the end, we pose a problem.

Open Problem. For positive real numbers x and y, and k holds

$$\frac{[\Gamma_k(x+y+1)/\Gamma_k(y+1)]^{\frac{1}{x}}}{[\Gamma_k(x+y+2)/\Gamma_k(y+1)]^{\frac{1}{x+1}}}<\sqrt{\frac{x+y}{x+y+1}}. \tag{20}$$

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Exponential stability of linear non-autonomous systems

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Abstract In this paper, we study the exponential stability of linear non-autonomous systems with multiple delays. Using Lyapunov-like function, we find sufficient conditions for the exponential stability in terms of the solution of a Riccati differential equation. Our results are illustrated with numerical examples.

Keywords Exponential stability, time-delay, Lyapunov function, Riccati equation.

§1. Introduction

The topic of Lyapunov stability of linear systems has been an interesting research area in the past decades. An integral part of the stability analysis of differential equations is the existence of inherent time delays. Time delays are frequently encountered in many physical and chemical processes as well as in the models of hereditary systems, Lotka-Volterra systems, control of the growth of global economy, control of epidemics, etc. Therefore, the stability problem of time-delay systems has been received considerable attention from many researchers (see e.g. [5, 6, 10, 12,14] and references therein). One of the extended stability properties is the concept of the α-stability, which relates to the exponential stability with a convergent rate $\alpha > 0$. Namely, a retarded system

$$\dot{x} = f(t, x(t), x(t-h)), \quad t \geq 0,$$
$$x(t) = \phi(t), \quad t \in [-h, 0],$$

is α-stable, with $\alpha > 0$, if there is a function $\xi(.)$ such that for each $\phi(.)$, the solution $x(t,\phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}, \quad \forall t \geq 0,$$

where $\|\phi\| = \max\{\|\phi(t)\| : t \in [-h, 0]\}$. This implies that for $\alpha > 0$, the system can be made exponentially stable with the convergent rate α. It is well known that there are many different methods to study the stability problem of time-delay linear autonomous systems. The widely used method is the approach of Lyapunov functions with Razumikhin techniques and the asymptotic stability conditions are presented in terms of the solution of either linear matrix inequalities or Riccati equations [2, 7, 8]. By using both the time-domain and the frequency-domain techniques, the paper [15] derived sufficient conditions for the asymptotic stability of a

linear autonomous system with multiple time delays of the form

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{i=1}^{m} A_i x(t-h_i), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{1}$$

where A_i are given constant matrices, $h = \max\{h_i : i = 1, 2, \cdots, m\}$. These conditions depend only on the eigenvalues of A_0 and the norm values of A_i of the system. For studying the α-stability problem, based on the asymptotic stability of the linear undelayed part, i.e. A_0 is a Hurwitz matrix, the papers [13, 14] proposed sufficient conditions for the α-stability of system (1) in terms of the solution of a scalar inequality involving the eigenvalues, the matrix measures and the spectral radius of the system matrices. It is worth noticing that although the approach used in these papers allows us to derive the less conservative stability conditions, but it can not be applied to non-autonomous delay systems. The reason is that, the assumption $A_0(t)$ to be a Hurwitz matrix for each $t \geq 0$, i.e. $\operatorname{Re}\lambda(A(t)) < 0$, for each t, does not implies the exponential stability of the linear non-autonomous system $\dot{x} = A_0(t)x$. It is the purpose of this paper to search sufficient conditions for the α-stability of non-autonomous delay systems. Using the Lyapunov-like function method, we develop the results obtained in [3, 14] to the non-autonomous systems with multiple delays. Do not using any Lyapunov stability theorem, we establish sufficient conditions for the α-stability of system (2), which are given in terms of the solution of a Riccati differential equation (RDE). These conditions do not involve any stability property of the system matrix $A_0(t)$. Although the problem of solving of RDEs is in general still not easy, various effective approaches for finding the solutions of RDEs can be found in [1, 4, 9, 16].

The paper is organized as follows. Section 2 presents notations, mathematical definitions and an auxiliary lemma used in the next section. The sufficient conditions for the α-stability are presented in Section 3. Numerical examples illustrated the obtained result are also given in Section 3. The paper ends with cited references.

§2. Preliminaries

The following notations will be used for the remaining this paper.

$\mathbb{R}^+$ denotes the set of all real non-negative numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $\langle ., . \rangle$ and the vector norm $\|.\|$;

$\mathbb{R}^{n\times r}$ denotes the space of all matrices of dimension $(n \times r)$. A^T denotes the transpose of the vector/matrix A; A matrix A is symmetric if $A = A^T$; I denotes the identity matrix;

$\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\operatorname{Re}\lambda : \lambda \in \lambda(A)\}$;

$\|A\|$ denotes the spectral norm of the matrix defined by

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)};$$

$\eta(A)$ denotes the matrix measure of the matrix A given by

$$\eta(A) = \frac{1}{2}\lambda_{\max}(A + A^T).$$

$C([a,b],\mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuous functions on $[a,b]$;
Matrix A is called semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in \mathbb{R}^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$;

In the sequel, sometimes for the sake of brevity, we will omit the arguments of matrix-valued functions, if it does not cause any confusion.

Let us consider the following linear non-autonomous system with multiple delays

$$\begin{aligned} \dot{x}(t) &= A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t-h_i), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-h, 0], \end{aligned} \tag{2}$$

where $h = \max\{h_i : i = 1, 2, \cdots, m\}, A_i(t), i = 0, 1, \cdots, m,$ are given matrix functions and $\phi(t) \in C([-h, 0], \mathbb{R}^n)$.

Definition. The system (2) is said to be α-stable, if there is a function $\xi(.) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $\phi(t) \in C([-h, 0], \mathbb{R}^n)$, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t,\phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+.$$

The following well-known lemma, which is derived from completing the square, will be used in the proof of our main result.

Lemma 2.1. Assume that $S \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Then for every $P, Q \in \mathbb{R}^{n\times n}$,

$$\langle Px, x\rangle + 2\langle Qy, x\rangle - \langle Sy, y\rangle \leq \langle (P + QS^{-1}Q^T)x, x\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

§3. Main results

Consider the linear non-autonomous delay system (2), where the matrix functions $A_i(t)$, $i = 0, 1, \cdots, m$, are continuous on $\mathbb{R}^+$. Let us set

$$A_{0,\alpha}(t) = A_0(t) + \alpha I, \quad A_{i,\alpha}(t) = e^{\alpha h_i} A_i(t), i = 1, 2, \cdots, m.$$

Theorem 3.1. The linear non-autonomous system (2) is α-stable if there is a symmetric semi-positive definite matrix $P(t)$, $t \in \mathbb{R}^+$ such that

$$\begin{aligned} &\dot{P}(t) + A_{0,\alpha}^T(t)[P(t) + I] + [P(t) + I]A_{0,\alpha}(t) \\ &+ \sum_{i=1}^{m}[P(t) + I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t) + I] + mI = 0. \end{aligned} \tag{3}$$

Proof. Let $P(t) \geq 0$, $t \in \mathbb{R}^+$ be a solution of the RDE (3). We take the following change of the state variable

$$y(t) = e^{\alpha t}x(t), \quad t \in \mathbb{R}^+,$$

then the linear delay system (2) is transformed to the delay system

$$\begin{aligned} \dot{y}(t) &= A_{0,\alpha}(t)y(t) + \sum_{i=1}^{m} A_{i,\alpha}(t)y(t-h_i), \\ y(t) &= e^{\alpha t}\phi(t), \quad t \in [-h, 0], \end{aligned} \tag{4}$$

Consider the following time-varying Lyapunov-like function

$$V(t,y(t)) = \langle P(t)y(t), y(t)\rangle + \|y(t)\|^2 + \sum_{i=1}^{m}\int_{t-h_i}^{t}\|y(s)\|^2ds.$$

Taking the derivative of $V(.)$ in t along the solution of $y(t)$ of system (4) and using the RDE (3), we have

$$\begin{aligned}
&\dot V(t,y(t))\\
&= \langle \dot P(t)y(t), y(t)\rangle + 2\langle P(t)\dot y(t), y(t)\rangle + 2\langle \dot y(t), y(t)\rangle + m\|y(t)\|^2 - \sum_{i=1}^{m}\|y(t-h_i)\|^2,\\
&= \langle \dot P(t)y(t), y(t)\rangle + 2\langle P(t)A_{0,\alpha}(t)y(t), y(t)\rangle + 2\sum_{i=1}^{m}\langle P(t)A_{i,\alpha}(t)y(t-h_i), y(t)\rangle\\
&\quad + 2\langle A_{0,\alpha}(t)y(t), y(t)\rangle + 2\sum_{i=1}^{m}\langle A_{i,\alpha}(t)y(t-h_i), y(t)\rangle\\
&\quad + m\|y(t)\|^2 - \sum_{i=1}^{m}\|y(t-h_i)\|^2,\\
&= \langle \dot P(t)y(t), y(t)\rangle + 2\langle (P(t)+I)A_{0,\alpha}(t)y(t), y(t)\rangle \qquad (5)\\
&\quad + 2\sum_{i=1}^{m}\langle (P(t)+I)A_{i,\alpha}(t)y(t-h_i), y(t)\rangle + m\|y(t)\|^2 - \sum_{i=1}^{m}\|y(t-h_i)\|^2,\\
&= -\sum_{i=1}^{m}\langle [P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I]y(t), y(t)\rangle\\
&\quad + 2\sum_{i=1}^{m}\langle [P(t)+I]A_{i,\alpha}(t)y(t-h_i), y(t)\rangle - \sum_{i=1}^{m}\langle y(t-h_i), y(t-h_i)\rangle\\
&= \sum_{i=1}^{m}\{-\langle [P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I]y(t), y(t)\rangle\\
&\quad + 2\langle [P(t)+I]A_{i,\alpha}(t)y(t-h_i), y(t)\rangle - \langle y(t-h_i), y(t-h_i)\rangle\}.
\end{aligned}$$

Applying Lemma 2.1 to the above equality, we have

$$\dot V(t,y(t)) \le 0, \quad \forall t \in \mathbb{R}^+.$$

Integrating both sides of this inequality from 0 to t, we find

$$V(t,y(t)) - V(0,y(0)) \le 0, \quad \forall t \in \mathbb{R}^+,$$

and hence

$$\langle P(t)y(t), y(t)\rangle + \|y(t)\|^2 + \sum_{i=1}^{m}\int_{t-h_i}^{t}\|y(s)\|^2ds \le \langle P_0 y(0), y(0)\rangle + \|y(0)\|^2 + \sum_{i=1}^{m}\int_{-h_i}^{0}\|y(s)\|^2ds,$$

where $P_0 = P(0) \ge 0$ is any initial condition. Since

$$\langle P(t)y, y\rangle \ge 0, \quad \int_{t-h_i}^{t}\|y(s)\|^2ds \ge 0,$$

$$\int_{-h_i}^{0}\|y(s)\|^2ds \le \|\phi\|\int_{-h_i}^{0}e^{\alpha s}ds = \frac{1}{\alpha}(1-e^{-\alpha h_i})\|\phi\|,$$

it follows that

$$\|y(t)\|^2 \leq \langle P_0 y(0), y(0)\rangle + \|y(0)\|^2 + \frac{1}{\alpha}\sum_{i=1}^{m}(1-e^{-\alpha h_i})\|\phi\|.$$

Therefore, the solution $y(t,\phi)$ of the system (4) is bounded. Returning to the solution $x(t,\phi)$ of system (2) and noting that

$$\|y(0)\| = \|x(0)\| = \phi(0) \leq \|\phi\|,$$

we have $\|x(t,\phi)\| \leq \xi(\|\phi\|)e^{-\alpha t}$ for all $t \in \mathbb{R}^+$, where

$$\xi(\|\phi\|) := \{\|P_0\|\|\phi\|^2 + \|\phi\|^2 + \frac{1}{\alpha}\sum_{i=1}^{m}(1-e^{-\alpha h_i})\|\phi\|\}^{\frac{1}{2}}.$$

This implies system (2) begin α-stable and completes the proof.

Remark. Note that the existence of a semi-positive definite matrix solution $P(t)$ of RDE (3) guarantees the boundedness of the solution of transformed system (4), and hence the exponential stability of the linear non-autonomous delay system (2). Also, the stability of $A(t)$ is not assumed.

Example 3.2. Consider the following linear non-autonomous delay system in $\mathbb{R}^2$:

$$\dot{x} = A_0(t)x + A_1(t)x(t-0.5) + A_2(t)x(t-1), \quad t \in \mathbb{R}^+,$$

with any initial function $\phi(t) \in C([-1,0],\mathbb{R}^2)$ and

$$A_0(t) = \begin{pmatrix} a_0(t) & 0 \\ 0 & -7.5 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} e^{-0.5}a_1(t) & 0 \\ 0 & e^{-0.5}\sqrt{3} \end{pmatrix},$$
$$A_2(t) = \begin{pmatrix} e^{-1}a_1(t) & 0 \\ 0 & e^{-1}\sqrt{3} \end{pmatrix},$$

where

$$a_0(t) = \frac{7e^{-9t}-5}{2(1+e^{-9t})}, \quad a_1(t) = \frac{1}{\sqrt{2}(1+e^{-9t})}.$$

We have $h_1 = 0.5$, $h_2 = 1$, $m = 2$ and the matrix $A_0(t)$ is not asymptotically stable, since $\operatorname{Re}\lambda(A(0)) = 0.5 > 0$. Taking $\alpha = 1$, we have

$$A_{0,\alpha}(t) = \begin{pmatrix} a_0(t)+1 & 0 \\ 0 & -6.5 \end{pmatrix}, \quad A_{1,\alpha}(t) = A_{2,\alpha}(t) = \begin{pmatrix} a_1(t) & 0 \\ 0 & \sqrt{3} \end{pmatrix}.$$

The solution of RDE (3) is

$$P(t) = \begin{pmatrix} e^{-9t} & 0 \\ 0 & 1 \end{pmatrix} \geq 0, \quad \forall t \in \mathbb{R}^+.$$

Therefore, the system is 1-stable.

For the autonomous delay systems, we have the following α-stability condition as a consequence.

Corollary 3.3. The linear delay system (2), where A_i are constant matrices, is α-stable if there is a symmetric semi-positive definite matrix $P \in \mathbb{R}^{n\times n}$, which is a solution of the algebraic Riccati equation

$$A_{0,\alpha}^T[P+I] + [P+I]A_{0,\alpha} + \sum_{i=1}^m [P+I]A_{i,\alpha}A_{i,\alpha}^T[P+I] + mI = 0. \tag{6}$$

Example 3.4. Consider the linear autonomous delay system

$$\dot{x}(t) = A_0x(t) + A_1x(t-2) + A_2x(t-4), \quad t \in \mathbb{R}^+,$$

with any initial function $\phi(t) \in C([-4,0],\mathbb{R}^2)$ and

$$A_0 = \begin{pmatrix} -\frac{17}{6} & 0 \\ \frac{4}{3} & -3.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} e^{-1} & 0 \\ 0 & e^{-1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} e^{-2} & 0 \\ 0 & e^{-2} \end{pmatrix}.$$

In this case, we have $m = 2$, $h_1 = 2$, $h_2 = 4$. Taking $\alpha = 0.5$, we find

$$A_{0,\alpha}(t) = \begin{pmatrix} -\frac{7}{3} & 0 \\ \frac{4}{3} & -3 \end{pmatrix}, \quad A_{1,\alpha}(t) = A_{2,\alpha}(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the solution of algebraic Riccati equation (6) is

$$P = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0.$$

Therefore, the system is 0.5-stable.

Remark. Note that we can estimate the value of $V(t,y)$ as follows. Since

$$2(P+I)A_{0,\alpha} = A_0^TP + PA_0 + A_0 + A_0^T + 2\alpha(P+I),$$

from (5) it follows that

$$\begin{aligned}\dot{V}(t,y(t)) &= \langle[\dot{P}(t) + A_0^T(t)P(t) + P(t)A_0(t) + mI]y(t), y(t)\rangle \\ &\quad + \langle[A_0(t) + A_0^T(t)]y(t), y(t)\rangle + 2\alpha\langle(P(t)+I)y(t), y(t)\rangle \\ &\quad + \sum_{i=1}^m \Big\{2\langle[P(t)+I]A_{i,\alpha}(t)y(t-h_i), y(t)\rangle - \|y(t-h_i)\|^2\Big\}.\end{aligned}$$

Using Lemma 2.1, we have

$$\begin{aligned}&\sum_{i=1}^m \Big\{2\langle[P+I]A_{i,\alpha}y(t-h_i), y(t)\rangle - \|y(t-h_i)\|^2\Big\} \\ &\leq \sum_{i=1}^m \langle[P+I]A_{i,\alpha}A_{i,\alpha}^T[P+I]y(t), y(t)\rangle.\end{aligned}$$

On the other hand, since

$$\sum_{i=1}^{m}\langle [P(t)+I]A_{i,\alpha}(t)A_{i,\alpha}^T(t)[P(t)+I]y(t),y(t)\rangle \le m\|P(t)+I\|^2e^{2\alpha h}\|A(t)\|^2\|y(t)\|^2,$$

with $h=\max\{h_1,h_2,\cdots,h_m\}$, $\|A(t)\|^2=\max\{\|A_1(t)\|^2,\|A_2(t)\|^2,\cdots,\|A_m(t)\|^2\}$, we obtain

$$\begin{aligned}\dot V(t,y(t)) &\le \langle [\dot P(t)+A_0^T(t)P(t)+P(t)A_0(t)+mI]y(t),y(t)\rangle\\ &\quad+\Big[2\eta(A_0(t))+2\alpha\|P(t)+I\|+m\|P(t)+I\|^2e^{2\alpha h}\|A(t)\|^2\Big]\|y(t)\|^2.\end{aligned}$$

Therefore, the α-stability condition of Theorem 3.1 can be given in terms of the solution of the following Lyapunov equation, which does not involve α:

$$\dot P(t)+A_0^T(t)P(t)+P(t)A_0(t)+mI=0. \tag{7}$$

In this case, if we assume that $P(t)$, $A_i(t)$ are bounded on $\mathbb{R}^+$ and

$$\eta(A_0):=\sup_{t\in\mathbb{R}^+}\eta(A_0(t))<+\infty, \tag{8}$$

then the rate of convergence $\alpha>0$ can be defined as a solution of the scalar inequality

$$\eta(A_0)+\alpha\|P_I\|+\frac{m}{2}e^{2\alpha h}\|P_I\|^2\|A\|^2\le 0, \tag{9}$$

where

$$P_I=\sup_{t\in\mathbb{R}^+}\|P(t)+I\|,\quad \|A\|^2=\sup_{t\in\mathbb{R}^+}\|A(t)\|^2.$$

Therefore, we have the following α-stability condition.

Theorem 3.5. Assume that the matrix functions $A_i(t), i=1,2,\cdots,m$ are bounded on $\mathbb{R}^+$ and the conditions (8), (9) hold. The non-autonomous delay system (2) is α-stable if the Lyapunov equation (7) has a solution $P(t)\ge 0$, which is bounded on $\mathbb{R}^+$. In this case, the rate of convergence $\alpha>0$ is the solution of the inequality (9).

Example 3.6. Consider the linear non-autonomous delay system

$$\dot x(t)=A_0(t)x(t)+A_1(t)x(t-0.5)+A_2(t)x(t-1),\quad t\in\mathbb{R}^+,$$

with any initial function $\phi(t)\in C([-1,0],\mathbb{R}^2)$ and

$$A_0(t)=\begin{pmatrix}0.5-e^t & 1\\ -1 & 0.5-e^t\end{pmatrix},\quad A_1(t)=e^{-0.2}\sin t\begin{pmatrix}\frac{1}{40} & 0\\ 0 & \frac{1}{40}\end{pmatrix},$$
$$A_2(t)=e^{-0.2}\cos t\begin{pmatrix}\frac{1}{40} & 0\\ 0 & \frac{1}{40}\end{pmatrix}.$$

We have $m=2$, $h_1=0.5$, $h_2=1$, $\eta(A_0)=-0.5$ and $\|A\|=e^{-0.2}/40$. On the other hand, the solution of Lyapunov equation (7) is

$$P(t)=\begin{pmatrix}e^{-t} & 0\\ 0 & e^{-t}\end{pmatrix},$$

and then $\|P_I\| = 2$. The rate of convergence found from inequality (9) is $\alpha = 0.2$. All conditions of Theorem 3.5 hold and hence the system is 0.2-stable.

For the autonomous case, Theorem 3.5 gives the following α-stability condition, which is similar to that obtained in [3, 14].

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On left-e *wrpp* semigroups [1]

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Abstract Let e be an idempotent of a semigroup S. A *wrpp* semigroup S is said to be left-e *wrpp* if $xey = exy$ for all $x, y \in S^1$ with $y \neq 1$. This kind of semigroups is a natural generalization of the known *C-wrpp* semigroups. In this paper, we prove that a semigroup S is left-e *wrpp* if and only if S is a spined product of an *C-wrpp* semigroup and a right normal band. Our results extend the structure of *rpp* semigroups with left central idempotents and also the structure of *C-wrpp* semigroups described by X. D. Tang in 1997.

Keywords *Wrpp* semigroups, left-e *wrpp* semigroups, spined product of semigroups.

§1. Introduction

In generalizing the regular semigroups, Tang [11] in 1997 introduced a generalized Green relation $\mathcal{L}^{**}$ on a semigroup S by defining $(a, b) \in \mathcal{L}^{**}$ for $a, b \in S$ such that $(ax, ay) \in \mathcal{R}$ if and only if $(bx, by) \in \mathcal{R}$ for any $x, y \in S^1$, where $\mathcal{R}$ is the usual Green relation. It is easy to observe that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \mathcal{L}^{**}$ and moreover, Tang [11] shown that $\mathcal{L}^{**}$ is a right congruence on any semigroup S. According to [2], we formulate the definition of *wrpp* semigroups as follows.

Definition 1.1. A semigroup S is called *wrpp* if the following conditions hold:

(i) Every $\mathcal{L}^{**}$-class of S contains an idempotent of S;

(ii) For all $e \in E(L_a^{**})$,$a = ae$, where L_a^{**} is the $\mathcal{L}^{**}$-class of S containing $a \in S$ and $E(L_a^{**})$ is the set of idempotents in L_a^{**}.

It is clear that a regular semigroup is a *wrpp* semigroup but not conversely. Hence, *wrpp* semigroups are generalizations of regular semigroups. The class of *wrpp* semigroups and some of its subclasses have recently been studied by Du-Shum and Ren-Shum (see, [1, 5, 6]). For the structure of rpp semigroups and abundant semigroups, the reader is referred to the survey articles [9] and [10].

In order to further generalize the *C-wrpp* semigroups, we consider the left-e *wrpp* semigroup.

Definition 1.2. A *wrpp* semigroup S is said to be a left-e *wrpp* semigroup if $xey = exy$ holds for all $x, y \in S^1$ and $y \neq 1$, where e is any element in the set $E(S)$, the set of all idempotents of S.

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It can be easily seen that the left-e *wrpp* semigroups are natural generalizations of the known rpp semigroups with central idempotents studied by Fountain in [2]. It is noted that the left-e *wrpp* semigroups that we considered are quite different from the left C-rpp semigroups described by Guo, Shum and Zhu in [3]. For more information of the structure of rpp semigroups and their generalized classes, the reader is refereed to the recent article by Shum in [10].

In this paper, we describe a left-e *wrpp* semigroup S and study the smallest *C-wrpp* semigroup congruence on such a semigroup. We prove that a semigroup S is a left-e *wrpp* semigroup if and only if S can be expressed as a spined product of an *C-wrpp* semigroup and a right normal band. Our result not only generalizes the structure theorem of *rpp* semigroups with left central idempotents in [6] and also the structure theorem of *C-wrpp* semigroups in [11].

For terminologies and notations not given in this paper, the reader is referred to Howie [4]. For the structure of regular semigroup and its generalized classes, the reader is refereed to Shum-Guo in [8].

§2. Basic results

We first establish the following lemmas of left-e *wrpp* semigroups:

Lemma 2.1. If S is a left-e *wrpp* semigroup, then every $\mathcal{L}^{**}$-class of S contains a unique idempotent.

Proof. Let $a \in S$. Then there exists $e \in E(L_a^{**})$ such that $a = ae$. By our hypothesis, $ea = eae = ae = a$. Hence, if $f \in E(L_a^{**})$, then it is clear that $(e, f) \in \mathcal{L}^{**}$. Consequently, $f = fe = ef = e$.

We now denote the unique idempotent in L_a^{**} of S by a^+. Since S is a left-e *wrpp* semigroup, $a^+a = a = aa^+$ for all $a \in S$.

Lemma 2.2. If S is a left-e *wrpp* semigroup, then $\mathcal{L}^{**}$ is a congruence on S.

Proof. Take $(a, b) \in \mathcal{L}^{**}$ for $a, b \in S$. Then, by Lemma 2.1, we have $a^+ = b^+$. If $(cax, cay) \in \mathcal{R}$, for all $x, y \in S^1$ and $c \in S$, then by $c\mathcal{L}^{**}c^+$, $(c^+ax, c^+ay) = (c^+aa^+x, c^+aa^+y) \in \mathcal{R}$ and hence $(ac^+a^+x, ac^+a^+y) \in \mathcal{R}$. By $a\mathcal{L}^{**}b$, we obtain $(bc^+a^+x, bc^+a^+y) \in \mathcal{R}$ so that $(c^+bb^+x, c^+bb^+y) = (c^+bx, c^+by) \in \mathcal{R}$ since $a^+ = b^+$. It is now clear from $c\mathcal{L}^{**}c^+$ that $(cbx, cby) \in \mathcal{R}$.

Similarly, $(cbx, cby) \in \mathcal{R}$ implies $(cax, cay) \in \mathcal{R}$, and hence $(ca, cb) \in \mathcal{L}^{**}$. This shows that $\mathcal{L}^{**}$ is a left congruence on S and so $\mathcal{L}^{**}$ is a congruence on S.

Lemma 2.3. Let S be a left-e *wrpp* semigroup. Then $(ab)^+ = a^+b^+$ for all $a, b \in S$.

Proof. It is clear that $a\mathcal{L}^{**}a^+$ and $b\mathcal{L}^{**}b^+$. Since $\mathcal{L}^{**}$ is a congruence on S, $ab\mathcal{L}^{**}a^+b^+$, for all $a, b \in S$. Hence $(ab)^+ = a^+b^+$ by Lemma 2.1.

Recall that a right normal band E is a semilattice Y of right zero bands $E_\alpha (\alpha \in Y)$ (see [3]). We now write the subsemigroup E_α as $E(e)$ when $e \in E_\alpha$. Also, we denote $E(e) \leq E(f)$ if $E(e)E(f) \subseteq E(e)$.

We now define the following right E-balanced relation γ on a left-e *wrpp* semigroup.

Definition 2.4. Let S be a left-e *wrpp* semigroup and let $a, b \in S$. Define γ by $a\gamma b$ if and only if $a = bf$, for some $f \in E(b^+)$. We call γ a right E-balanced relation on S.

Lemma 2.5. Let γ be a right E-balanced relation on a left-e *wrpp* semigroup S. Then γ is a congruence on S.

Proof. For any $a, b \in S$, we first claim that if $a\gamma b$ then $E(a^+) = E(b^+)$. To prove our claim, suppose that $a\gamma b$. Then by the definition of γ, $a = bf$ for some $f \in E(b^+)$. It is clear that $af = a$. By Lemma 2.3, $a^+ = (af)^+ = a^+f$. This implies that $E(a^+)E(f) \subseteq E(a^+)$ which gives $E(a^+) \leq E(f) = E(b^+)$. Since $a = bf$ for some $f \in E(b^+)$ and $E(b^+)$ is a right zero band, we have $b = bb^+ = b(fb^+) = ab^+$. By Lemma 2.3, $b^+ = a^+b^+$ implies that $E(b^+) \leq E(a^+)$ and so $E(a^+) = E(b^+)$. Thus our claim is proved.

We now show that γ is an equivalent relation. The reflexivity and symmetry is clear. To prove that γ is transitive, we suppose that $a\gamma b$ and $b\gamma c$. Then, we have $E(a^+) = E(b^+) = E(c^+)$. By the definition of γ, there exist some $f \in E(b^+)(= E(c^+))$ and $g \in E(c^+)$ such that $a = bf$ and $b = cg$. It hence follows that $a = (cg)f = c(gf) = cf$ since $E(c^+)$ is a right zero band. Consequently, $a\gamma c$ and hence γ is an equivalent relation on S. Finally, we show that γ is compatible with semigroup multiplication. To prove that γ is left compatible with semigroup multiplication on S, that is, to prove that γ is a left congruence on S, we assume that $a\gamma b$ for $a, b \in S$. Then $a = bf$ for some $f \in E(b^+)$. Clearly $ca = cbf = cb(cb)^+f$. Since $cb = cbb^+$ for any $c \in S$ and by Lemma 2.3, $(cb)^+ = (cb)^+b^+$ holds. This leads to $E[(cb)^+] = E[(cb)^+b^+] = E[(cb)^+f]$ and so $(cb)^+f \in E[(cb)^+]$.Thus,$ca\gamma cb$. Since every idempotent of S is left central, it is easy to verify that γ is also a right congruence on S. Hence, γ is indeed a congruence on S.

Lemma 2.6. Let S be a left-e *wrpp* semigroup. If $a\mathcal{L}^{**}b$ for $a, b \in S$ then $a\gamma\mathcal{L}^{**S/\gamma}b\gamma$.

Proof. We first prove that if $a\mathcal{L}^{**}b$, then $((ax)\gamma, (ay)\gamma) \in \mathcal{R}^{S/\gamma}$ implies $((bx)\gamma, (by)\gamma) \in \mathcal{R}^{S/\gamma}$, for any $x, y \in S^1$ and $x\gamma, y\gamma \in (S/\gamma)^1$.

Assume that $((ax)\gamma, (ay)\gamma) \in \mathcal{R}^{S/\gamma}$. Then there exist u, v in S^1 such that $(ax)\gamma u\gamma = (ay)\gamma$ and $(ay)\gamma v\gamma = (ax)\gamma$, that is, $(axu)\gamma = (ay)\gamma$ and $(ayv)\gamma = (ax)\gamma$. Now, by the definition of γ, we have $ay = (axu)e = ax(ue)$, for some $e \in E[(axu)^+]$ and $ax = (ayv)f = ay(vf)$, for some $f \in E[(ayv)^+]$. These equalities imply $(ax, ay) \in \mathcal{R}$ and so $(bx, by) \in \mathcal{R}$ since $a\mathcal{L}^{**}b$. Hence there exist $s, t \in S^1$ such that $bx = bys$ and $by = bxt$. It can be immediately seen that $((bx)\gamma, (by)\gamma) \in \mathcal{R}^{S/\gamma}$.

Similarly, $((bx)\gamma, (by)\gamma) \in \mathcal{R}^{S/\gamma}$ implies $((ax)\gamma, (ay)\gamma) \in \mathcal{R}^{S/\gamma}$. Thus, $a\gamma\mathcal{L}^{**S/\gamma}b\gamma$.

Lemma 2.7. If S is a left-e *wrpp* semigroup, then $E(S/\gamma) = \{x \in S/\gamma \mid (\exists e \in E(S)) \quad e\gamma = x\}$, where γ is the congruence given in Lemma 2.5 on S.

Proof. Let $\overline{E} = \{x \mid (\exists e \in E(S))e\gamma = x\}$. Then, it is clear that $\overline{E} \subseteq E(S/\gamma)$. Conversely, if $x\gamma \in E(S/\gamma)$, then $(x\gamma)^2 = x\gamma$. Then, by the definition of γ, we have $x^2 = xf$, for some $f \in E(x^+)$. This leads to $x^2 = x(xx^+) = x^2x^+ = xfx^+ = xx^+ = x$ and hence $E(S/\gamma) \subseteq \overline{E}$.

We now formulate the following theorem.

Theorem 2.8. Suppose that S is a left-e *wrpp* semigroup. Then S/γ is the maximum homomorphism image of S such that S/γ is an *C-wrpp* semigroup.

Proof. By our assumption and Lemma 2.6, S/γ is a *wrpp* semigroup. To see that S/γ is an *C-wrpp* semigroup, we need to prove that every idempotent of S/γ lies in the center of S/γ. Put $x\gamma \in S/\gamma$ and $y\gamma \in E(S/\gamma)$. Then by Lemma 2.7, there exists an idempotent e of S such that $e\gamma = y\gamma$. Thus, we deduce that

$$xe = exe = ex(x^+e).$$

On the other hand, it can be verified that $(x^+e)(ex^+) = ex^+$ and $(ex^+)(x^+e) = x^+e$. This implies that $x^+e \in E(ex^+) = E[(ex)^+]$. Consequently, $(xe, ex) \in \gamma$, that is, $(x\gamma)(y\gamma) = (x\gamma)(e\gamma) = (e\gamma)(x\gamma) = (y\gamma)(x\gamma)$. Thus, S/γ is an *C-wrpp* semigroup.

We now proceed to prove that γ is the smallest *C-wrpp* congruence on S. Let ρ be a congruence on S such that S/ρ is an *C-wrpp* semigroup, and suppose that $(a, b) \in \gamma$. Then $a = bf$ for some $f \in E(b^+)$. Since S/ρ is an *C-wrpp* semigroup whose idempotents are central and $E(b^+)$ is a right zero band, it follows that $a\rho = b\rho f\rho = f\rho b\rho = f\rho(b^+b)\rho = b^+\rho b\rho = b\rho$, and consequently $\gamma \subseteq \rho$.

§3. Structure theorem

We now establish a structure theorem for a left-e *wrpp* semigroup.

Recall that a semigroup S is $\mathcal{R}$-left cancellative if $(ab, ac) \in \mathcal{R}$ implies $(b, c) \in \mathcal{R}$, for all $a, b, c \in S$. It was proved in [11] that a semigroup S is an *C-wrpp* semigroup if and only if S is a strong semilattice of $\mathcal{R}$-left cancellative monoids.

For the sake of brevity, we call a semigroup S an $\mathcal{R}$-left cancellative plank if S is the direct product of an $\mathcal{R}$-left cancellative monoid and a right zero band. In view of this terminology, we have the following lemma.

Theorem 3.1. The following conditions on a semigroup S are equivalent:

(i) S is a left-e *wrpp* semigroup.

(ii) S is a spined product of an *C-wrpp* semigroup and a right normal band with respect to a semilattice Y.

(iii) S is a strong semilattice of $\mathcal{R}$-left cancellative planks.

Proof. (i)⇒(ii) Let S be a left-e *wrpp* semigroup. Then by Theorem 2.8, S/γ is an *C-wrpp* semigroup. According to Tang [9], S/γ can be expressed as a strong semilattice $[Y; M_\alpha, \phi_{\alpha,\beta}]$ of $\mathcal{R}$-left cancellative monoids $M_\alpha(\alpha \in Y)$, where M_α are $\mathcal{L}^{**}$-classes of S/γ, $Y = (S/\gamma)/\mathcal{L}^{**}$. It is clear that $\gamma\,|_{E(S)}= \mathcal{J}^{E(S)}$. Since $E(S)$ is a right normal band, $E(S) = [Y; E_\alpha, \varphi_{\alpha,\beta}]$ where $Y = E(S)/\gamma|_{E(S)}$ and E_α are right zero bands. Now we form the spined product of S/γ and $E(S)$ with respect to Y, denoted by $T = \bigcup_{\alpha\in Y}(M_\alpha \times E_\alpha)$, where the multiplication on T is defined by $(m, i) \cdot (n, j) = (mn, ij)$. In the above multiplication, mn, ij are the semigroup products of m, n in S/γ and i, j in $E(S)$, respectively. To show that $S \simeq T$, we consider the following mapping $\theta : S \longrightarrow T$ by $s \longmapsto (s\gamma, s^+)$.

We first show that θ is injective. For this purpose, let $(s\gamma, s^+) = (t\gamma, t^+)$, for $(s\gamma, s^+)$, $(t\gamma, t^+) \in T$. Clearly, $s\gamma = t\gamma$ and $s^+ = t^+$. By the definition of γ, there exists $f \in E(t^+)$ such that $s = tf$. Since $E(t^+)$ is a right zero band, $s = ss^+ = tft^+ = tt^+ = t$ and hence θ is an injective mapping.

To show that θ is a surjection, pick $(a, i) \in M_\alpha \times E_\alpha \subseteq T$ for some $\alpha \in Y$. Then there exists $x \in S$ such that $x\gamma = a \in M_\alpha$ and $x^+ \in E_\alpha$. Clearly, $i \in E_\alpha$. Since x^+, i are elements of E_α, it follows that $(xi)\gamma = (xx^+)\gamma = x\gamma = a$. On the other hand, by Lemma 2.3, $(xi)^+ = x^+i = i$. This shows that $(xi)\theta = (a, i)$, and thereby θ is a surjective mapping.

Finally, we prove that θ is a homomorphism from S to T. By using Lemma 2.3 again, we

can easily deduce the following

$$
\begin{aligned}
(st)\theta &= ((st)\gamma, (st)^+) = (s\gamma t\gamma, s^+t^+) \\
&= (s\gamma, s^+)(t\gamma, t^+) = (s\theta)(t\theta).
\end{aligned}
$$

This shows that θ is an isomorphism. Thus, S is isomorphic to the spined product of an C-*wrpp* semigroup and a right normal band.

(ii)⇒(iii) Suppose that $S = M\Pi_{(Y,\xi,\eta)}E$ is a spined product of an C-*wrpp* semigroup M and a right normal band E with respect to a semilattice Y. It is clear that $M = [Y; M_\alpha, \phi_{\alpha,\beta}]$ and $E = [Y; E_\alpha, \psi_{\alpha,\beta}]$. Then, for all $\alpha, \beta \in Y$ with $\alpha \geq \beta$ and $(a, i) \in M_\alpha \times E_\alpha$, define a mapping $\Phi_{\alpha,\beta}$ from S to S by :

$$(a, i)\Phi_{\alpha,\beta} = (a\phi_{\alpha,\beta}, i\psi_{\alpha,\beta}),$$

where $\phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ are structure homomorphisms for M and E, respectively. Obviously, $\Phi_{\alpha,\beta}$ is well-defined. It is easy to see that $\Phi_{\alpha,\beta}$ is a homomorphism. Also, it is trivial that $\Phi_{\alpha,\beta}\Phi_{\beta,\gamma} = \Phi_{\alpha,\gamma}$, for all α,β and γ in Y with $\alpha \geq \beta \geq \gamma$. Now, we let $(x, k) \in M_\alpha \times E_\alpha$ and $(y, j) \in M_\beta \times E_\beta$, and write $\gamma = \alpha\beta$. Then it follows that

$$(x, k)(y, j) = (xy, kj) = (x, k)\Phi_{\alpha,\gamma}(y, j)\Phi_{\beta,\gamma}.$$

Thus, S is isomorphic to a strong semilattice of $\mathcal{R}$-left cancellative planks $M_\alpha \times E_\alpha$.

(iii)⇒(i) Suppose that $S = [Y; M_\alpha \times E_\alpha, \Phi_{\alpha,\beta}]$ is a strong semilattice of $\mathcal{R}$-left cancellative planks $M_\alpha \times E_\alpha$ with the structure morphism $\Phi_{\alpha,\beta}$. For brevity, we write $S_\alpha = M_\alpha \times E_\alpha$.

Firstly we show that $E(S) = \bigcup_{\alpha\in Y}\{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in E_\alpha\}$. Let $(a, i) \in E(S)$. Then there exists $\alpha \in Y$ such that $(a, i) \in E(S_\alpha)$ and $(a, i)^2 = (a^2, i) = (a, i)$. Clearly, $a^2 = a$ and $(a, a^2) \in \mathcal{R}(M_\alpha)$. Since M_α is an $\mathcal{R}$-left cancellative monoid, $(1_\alpha, a) \in \mathcal{R}(M_\alpha)$. Thus, there exists $u \in M_\alpha^1$ such that $1_\alpha = au$. Consequently, it follows that

$$a = a1_\alpha = a \cdot au = a^2u = au = 1_\alpha.$$

This shows that $E(S) \subseteq \bigcup_{\alpha\in Y}\{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in E_\alpha\}$. The converse inclusion is immediate. Now, let

$$E(S) = \bigcup_{\alpha\in Y}\{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in E_\alpha\}.$$

Then we claim that every idempotent of S is left central. For this purpose, let $a, b \in S^1, b \neq 1$ and $e \in E(S)$. Then there exist $\alpha, \beta, \gamma \in Y$ such that $a \in S_\alpha^1, b \in S_\beta$ and $e \in E(S_\gamma)$. Write $\delta = \alpha\beta\gamma, a\Phi_{\alpha,\delta} = (x, i) \in S_\delta, b\Phi_{\beta,\delta} = (y, j) \in S_\delta$ and $e\Phi_{\gamma,\delta} = (1_\delta, k) \in E(S_\delta)$. Thus, we have

$$aeb = (a\Phi_{\alpha,\delta})(e\Phi_{\gamma,\delta})(b\Phi_{\beta,\delta}) = (xy, j).$$

Similarly, $eab = (xy, j)$ and so $eab = aeb$.

We need to prove that S is a *wrpp* semigroup. Let $a = (x, i) \in S_\alpha$. Then $e = (1_\alpha, i) \in E(S_\alpha)$, and hence $ae = (x, i)(1_\alpha, i) = (x, i) = a$. For any $b \in S_\beta^1$ and $c \in S_\gamma^1$, if $(ab, ac) \in \mathcal{R}$, then $\alpha\beta = \alpha\gamma$. Furthermore, since $M_{\alpha\beta}$ is an $\mathcal{R}$-left cancellative monoid, $(eb, ec) \in \mathcal{R}$. Conversely,

if $(eb, ec) \in \mathcal{R}$, then $(aeb, aec) \in \mathcal{R}$, and hence $(ab, ac) \in \mathcal{R}$. Thus $a\mathcal{L}^{**}e$, and therefore S is a *wrpp* semigroup.

If S is a *rpp* semigroup such that $\mathcal{L}^* = \mathcal{L}^{**}$ holds on S, then we immediately re-obtain the following result in [6].

Corollary 3.2. [6] The following conditions on a semigroup S are equivalent:

(i) S is a *rpp* semigroup with left central idempotents;

(ii) S is a spined product of an C-*rpp* semigroup and a right normal band with respect to a semilattice Y;

(iii) S is a strong semilattice of left cancellative planks.

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On quasi b-open functions and quasi b-closed functions

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Abstract The purpose of this paper is to give two new type of functions called, quasi b-open functions and quasi b-closed functions. Also, we obtain its characterizations and its basic properties.

Keywords Topological spaces, quasi b-open functions, quasi b-closed functions.

§1. Introduction and preliminaries

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one considers openness. Its importance is significant in various areas of mathematics and related sciences. Since 1996, when Andrijevic [1] has introduced a weak form of open sets called b-open sets. In the same year, this notion was also called sp-open sets in the sense of Dontchev and Przemski [2] but one year later are called γ-open sets due to El-Atik [5]. In this paper, we will continue the study of related functions by involving b-open sets. We introduce and characterize the concept of quasi-b-open functions and quasi-b-closed functions in topological space. Throughout this paper, spaces means topological spaces on which no separation axioms are assumed unless otherwise mentioned and $f : (X, \tau) \to (Y, \sigma)$ (or simply $f : X \to Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a space (X, τ) is called b-open [1] (= sp-open [2], γ-open [5]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A)) \cup \mathrm{Int}(\mathrm{Cl}(A))$. The complement of a b-open set is called b-closed. The union (resp. intersection) of all b-open (resp. b-closed) sets, each contained in (resp. containing) a set A in a space X is called the b-interior (resp. b-closure) of A and is denoted by b-$\mathrm{Int}(A)$ (resp. b-$\mathrm{Cl}(A)$) [1].

§2. Quasi b-open functions

Definition 1. A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(i) b-irresolute [5] (b-continuous [5]) if $f^{-1}(V)$ is b-closed in X for every b-closed (resp. closed) subset V of Y;

(ii) b-open [5] (resp. b-closed [5]) if $f(V)$ is b-open (resp. b-closed) in Y for every open (resp. closed) subset of X.

Definition 2. A function $f : X \to Y$ is said to be quasi b-open if the image of every b-open set in X is open in Y.

It is evident that, the concepts quasi b-openness and b-continuity coincide if the function is a bijection.

Theorem 3. A function $f : X \to Y$ is quasi-b-open if and only if for every subset U of X, $f(b\text{-Int}(U)) \subset \text{Int}(f(U))$.

Proof of Theorem 3. Let f be quasi b-open function. Now, we have $\text{Int}(U) \subset U$ and $b\text{-Int}(U)$ is a b-open set. Hence, we obtain that $f(b\text{-Int}(U)) \subset f(U)$. As $f(b\text{-Int}(U))$ is open, then $f(b\text{-Int}(U)) \subset \text{Int}(f(U))$. Conversely, assume that U be a b-open set in X. Then, $f(U) = f(b\text{-Int}(U)) \subset \text{Int}(f(U))$ but usually $\text{Int}(f(U)) \subset f(U)$. Consequently, $f(U) = \text{Int}(f(U))$ and hence f is quasi b-open.

Lemma 4. A function $f : X \to Y$ is quasi b-open, then $b\text{-Int}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G))$ for every subset G of Y.

Proof of Lemma 4. Let G be any arbitrary subset of Y. Then, $b\text{-Int}(f^{-1}(G))$ is a b-open set in X and f is quasi b-open, then $f(b\text{-Int}(f^{-1}(G))) \subset \text{Int}(f(f^{-1}(G))) \subset \text{Int}(G)$. Thus, $b\text{-Int}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G))$.

Recall that a subset S is called a b-neighbourhood [5] of a point x of X if there exists a b-open set U such that $x \in U \subset S$.

Theorem 5. For a function $f : X \to Y$, the following are equivalent:

(i) f is quasi b-open;

(ii) For each subset U of X, $f(b\text{-Int}(U)) \subset \text{Int}(f(U))$;

(iii) For each $x \in X$ and each b-neighbourhood U of x in X, there exists a neighbourhood V of $f(x)$ in Y such that $V \subset f(U)$.

Proof of Theorem 5. (i)$\Rightarrow$(ii): It follows from Theorem 3.
(ii)$\Rightarrow$(iii): Let $x \in X$ and U be an arbitrary b-neighbourhood of x in X. Then there exists a b-open set V in X such that $x \in V \subset U$. Then by (ii), we have $f(V) = f(b\text{-Int}(V)) \subset \text{Int}(f(U))$ and hence $f(V) = \text{Int}(f(V))$. Therefore, it follows that $f(V)$ is open in Y such that $f(x) \in f(V) \subset f(U)$.
(iii)$\Rightarrow$(i): Let U be an arbitrary b-open set in X. Then for each $y \in f(U)$, by (iii) there exists a neighbourhood V_y of y in Y such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exists an open set W_y in Y such that $y \in W_y \subset V_y$. Thus, $f(U) = \bigcup\{W_y : y \in f(U)\}$ which is an open set in Y. This implies that f is quasi b-open function.

Theorem 6. A function $f : X \to Y$ is quasi b-open if and only if for any subset B of Y and for any b-closed set F of X containing $f^{-1}(B)$, there exists a closed set G of Y containing B such that $f^{-1}(G) \subset F$.

Proof of Theorem 6. Suppose f is quasi b-open. Let $B \subset Y$ and F be a b-closed set of X containing $f^{-1}(B)$. Now, put $G = Y \backslash f(X \backslash F)$. It is clear that $f^{-1}(B) \subset F$ implies $B \subset G$. Since f is quasi b-open, we obtain G as a closed set of Y. Then, we have $f^{-1}(G) \subset F$. Conversely, let U be a b-open set of X and put $B = Y \backslash f(U)$. Then $X - U$ is a b-closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset X \backslash U$. Hence, we obtain $f(U) \subset Y \backslash F$. On the other hand, it follows that $B \subset F$, $Y \backslash F \subset Y \backslash B = f(U)$. Thus, we obtain $f(U) = Y \backslash F$ which is open and hence f is a quasi b-open function.

Theorem 7. A function $f : X \to Y$ is quasi b-open if and only if $f(\mathrm{Cl}(B)) \subset b\text{-}\mathrm{Cl}(f(B))$ for every subset B of Y.

Proof of Theorem 7. Suppose that f is quasi b-open. For any subset B of Y, $f^{-1}(B) \subset b\text{-}\mathrm{Cl}(f^{-1}(B))$. Therefore by Theorem 6, there exists a closed set F in Y such that $B \subset F$ and $f^{-1}(F) \subset b\text{-}\mathrm{Cl}(f^{-1}(B))$. Therefore, we obtain $f^{-1}(\mathrm{Cl}(B)) \subset f^{-1}(F) \subset b\text{-}\mathrm{Cl}(f^{-1}(B))$. Conversely, let $B \subset Y$ and F be a b-closed set of X containing $f^{-1}(B)$. Put $W = \mathrm{Cl}_Y(B)$, then we have $B \subset W$ and W is closed and $f^{-1}(W) \subset b\text{-}\mathrm{Cl}(f^{-1}(B)) \subset F$. Then by Theorem 6, f is quasi b-open.

Lemma 8. Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $g \circ f : X \to Z$ is quasi b-open. If g is continuous injective, then f is quasi b-open.

Proof of Lemma 8. Let U be a b-open set in X, then $(g \circ f)(U)$ is open in Z since $g \circ f$ is quasi b-open. Again g is an injective continuous function, $f(U) = g^{-1}(g \circ f(U))$ is open in Y. This shows that f is quasi b-open.

§3. Quasi b-closed functions

Definition 1. A function $f : X \to Y$ is said to be quasi b-closed if the image of each b-closed set in X is closed in Y.

Clearly, every quasi b-closed function is closed as well as b-closed.

Remark 2. Every b-closed (resp. closed) function need not be quasi b-closed as shown by the following example.

Example 3. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is b-closed as well as closed but not quasi b-closed.

Lemma 4. A function $f : X \to Y$ is quasi b-closed if and only if $f^{-1}(\mathrm{Int}(B)) \subset b\text{-}\mathrm{Int}(f^{-1}(B))$ for every subset B of Y.

Proof of Lemma 4. This proof is similar to the proof of Lemma 4.

Theorem 5. A function $f : X \to Y$ is quasi b-closed if and only if for any subset B of Y and for any b-open set G of X containing $f^{-1}(B)$, there exists an open set U of Y containing B such that $f^{-1}(U) \subset G$.

Proof of Theorem 5. This proof is similar to that of Theorem 6.

Definition 6. A function $f : X \to Y$ is called pre-b-closed (γ-closed [3]) if the image of every b-closed subset of X is b-closed in Y.

Definition 7. A space X is said to be a T_b-space if every b-closed set in X is closed in X.

Remark 8. Let $f : X \to Y$ be a quasi b-closed function. If Y is a T_b-space, then quasi b-closedness coincide with pre-b-closedness.

Theorem 9. If $f : X \to Y$ and $g : Y \to Z$ are two quasi b-closed function, then $g \circ f$: $X \to Z$ is a quasi b-closed function.

Proof of Theorem 9. Obvious.

Furthermore, we have the following

Theorem 10. Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then

(i) If f is b-closed and g is quasi b-closed, then $g \circ f$ is closed;

(ii) If f is quasi b-closed and g is b-closed, then $g \circ f$ is pre b-closed;

(iii) If f is pre b-closed and g is quasi b-closed, then $g \circ f$ is quasi b-closed.

Proof of Theorem 10. Obvious.

Theorem 11. Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is quasi b-closed.

(i) If f is b-irresolute surjective, then g is closed.

(ii) If g is b-continuous injective, then f is pre b-closed.

Proof of Theorem 11. (i) Suppose F is an arbitrary b-closed set in Y. As f is b-irresolute, $f^{-1}(F)$ is b-closed in X. Since $g \circ f$ is quasi b-closed and f is surjective, $(g \circ f(f^{-1}(F))) = g(F)$, which is closed in Z. This implies that g is a closed function.
(ii) Suppose F is any b-closed set in X. Since $g \circ f$ is quasi b-closed, $(g \circ f)(F)$ is closed in Z. Again g is a b-continuous injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is b-closed in Y. This shows that f is pre b-closed.

Theorem 12. Let X and Y be topological spaces. Then the function $g : X \to Y$ is a quasi b-closed if and only $g(X)$ is closed in Y and $g(V)\backslash g(X\backslash V)$ is open in $g(X)$ whenever V is b-open in X.

Proof of Theorem 12. Suppose $g : X \to Y$ is a quasi b-closed function. Since X is b-closed, $g(X)$ is closed in Y and $g(V)\backslash g(X\backslash V) = g(X)\backslash g(X\backslash V)$ is open in $g(X)$ when V is b-open in X. Conversely, Suppose $g(X)$ is closed in Y, $g(V)\backslash g(X\backslash V)$ is open in $g(X)$ when V is b-open in X, and let C be closed in X. Then $g(C) = g(X)\backslash(g(X\backslash C)\backslash g(C))$ is closed in $g(X)$ and hence, closed in Y.

Corollary 13. Let X and Y be topological spaces. Then a surjective function $g : X \to Y$ is quasi b-closed if and only if $g(V)\backslash g(X\backslash V)$ is open in Y whenever U is b-open in X.

Proof of Corollary 13. Obvious.

Corollary 14. Let X and Y be topological spaces and let $g : X \to Y$ be a b-continuous quasi b-closed surjective function. Then the topology on Y is $\{g(V)\backslash g(X\backslash V) : V$ is b-open in $X\}$.

Proof of Corollary 14. Let W be open in Y. Then $g^{-1}(W)$ is b-open in X, and $g(g^{-1}(W)) \setminus g(X \setminus g^{-1}(W)) = W$. Hence, all open sets in Y are of the form $g(V)\backslash g(X\backslash V)$, V is b-open in X. On the other hand, all sets of the form $g(V)\backslash g(X\backslash V)$, V is b-open in X, are open in Y from Corollary 13.

Definition 15. A topological space (X, τ) is said to be b-normal (= γ-normal [4]) if for any pair of disjoint b-closed subsets F_1 and F_2 of X, there exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Theorem 16. Let X and Y be topological spaces with X is b-normal and let $g : X \to Y$ be a b-continous quasi b-closed surjective function. Then Y is normal.

Proof of Theorem 16. Let K and M be disjoint closed subsets of Y. Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint b-closed subsets of X. Since X is b-normal, there exist disjoint open sets V and W such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V)\backslash g(X\backslash V)$ and $M \subset g(W)\backslash g(X\backslash W)$. Further by Corollary 13, $g(V)\backslash g(X\backslash V)$ and $g(W)\backslash g(X - W)$ are open sets in Y and clearly $(g(V)\backslash g(X\backslash V)) \bigcap (g(W)\ g(X\backslash W)) = \varnothing$. This shows that Y is normal.

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Bayesian analysis for a mixture of log-normal distributions

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Abstract Mixed distributions play an important role in lifetime data analysis. In this paper we consider parameters estimation of a mixture for log-normal distributions by means of gibbsing sampling algorithm under the squared errors loss function. Some simulations show that bayesian algorithm is effective to our model.

Keywords Mixed log-normal distributions, bayesian analysis, gibbs sampling algorithm, squared errors loss function.

§1. Introduction

Mixted models have been paid great attention in many applicable fields, such as medicine analyis, psychology research, cluster analysis, life testing and reliability analysis and so on in past 30 years, discussed by Everitt and Hand (1981), Mclachlan and Peel (2000) etc. In general, we may estimate mixture models by means of the EM algorithm and bayesian method. The EM algorithm has been extensively used to estimate mixture models, see Sultan, Ismail and Al-Moisheer (2007), Jiang, Murthy, Ji (2007), Elsherpieny (2007). Bayesian estimate for mixture models have been also developed enormously because of great progress of bayesian computation in recent 20 years, see Diebolt and Robert (1994), Bauwens and Rombouts (2007), Bauwens, Hafner and Rombouts (2007), Anduin (2009), Bougulia, Ziou, Hammoud (2009) and so on. Log-normal distribution has quite wide-ranging use in economics, finance and insurance, reliability analysis etc. This paper will consider parameters estimation of mixed log-normal distributions by means of gibbs sampling algorithm in the bayesian framework. The mixture of log-normal distributions (MLND) has its probability density function as

$$f(x|p,\mu,\theta)=\sum_{i=1}^{m}p_i\cdot\frac{1}{\sqrt{2\pi\theta_i}x}exp\{-\frac{(lnx-\mu_i)^2}{2\theta_i}\},\qquad x>0 \tag{1}$$

where $p=(p_1,\cdots,p_m)^T, 0<p_i<1, i=1,\cdots,m-1; p_m=1-\sum_{i=1}^{m-1}p_i; \mu=(\mu_1,\cdots,\mu_m)^T; \theta=(\theta_1,\cdots,\theta_m)^T$, we impose the identifying restriction condition on model (1) $0<\theta_1<\theta_2<\cdots<$

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θ_m, and there are $3m-1$ parameters in all. The remainder of this paper has the following organization. In section 2, we consider parameters estimation of the MLND given in model (1) by mean of gibbs sampling algorithm. In section 3, some simulations are carried out to illustrate the estimation effect. In the last section, we draw some conclusion about this paper.

§2. Parameter estimation by gibbs sampling algorithm

Suppose $X = (x_1, \cdots, x_N)$ is the observed data vector form the MLND given in (1), we denote

$$f_{ij} = \frac{1}{\sqrt{2\pi\theta_i}x_j} exp\{-\frac{(lnx_j - \mu_i)^2}{2\theta_i}\},\ i = 1, \cdots, m, j = 1, \cdots, N$$

$$f_j = \sum_{i=1}^{m} p_i f_{ij},\ j = 1, \cdots, N$$

We introduce a so-called state variable $S_j \in \{1, 2, \cdots, m\}$ for each observation that takes the value k when the observation come from component k. In like manner, we denote $S^N = (S_1, S_2, \cdots, S_N)$ that stands for a state vector of all N observations. The model specification assumes that the state variables are independent given the group probabilities denoted by $p = (p_1, \cdots, p_m)^T$, where p_k is the group k probability, namely, $p_k = P(S_j = k|p)$. Thereby, we can obtain the joint density of the state vector S^N given the model parameters is

$$\varphi(S^N|p) = \prod_{j=1}^{N} \varphi(S_j|p) = \prod_{j=1}^{N} p_{S_j}.$$

When given S^N and X, the likelihood function of the MLND model is

$$\varphi(p, \mu, \theta|S^N, X) = \prod_{j=1}^{N} p_{S_j} f_{S_j j} = \prod_{j=1}^{N} p_{S_j} \frac{1}{\sqrt{2\pi\theta_{S_j}}x_j} exp\{-\frac{(lnx_j - \mu_{S_j})^2}{2\theta_{S_j}}\}.$$

Since the state vector S^N can't be observed, we treat S^N as a parameter or random vector in bayesian framework which is called data augmentation method that makes inference more easy in despite of more parameters are been introduced. In the above-mentioned conditions, we obtain the joint posterior distribution of all parameters is given by

$$\varphi(p, \mu, \theta, S^N|X) \propto \varphi(p) \cdot \varphi(\mu) \cdot \varphi(\theta) \cdot \varphi(p, \mu, \theta|S^N, X), \tag{2}$$

where $\varphi(p), \varphi(\mu), \varphi(\theta)$ are the prior densities of the parameters p, μ, θ. And we suppose prior independence between p, μ, θ. Following in order, we will give the full conditional posterior densities of the different parameter blocks.

§2.1. Sampling S^N from $\varphi(S^N|p, \mu, \theta, X)$

Given p, μ, θ, X, the posterior density of S^N is in proportion to $\varphi(S^N|p) \cdot \varphi(p, \mu, \theta|S^N, X)$. Because of the independence of all coordinates each other of the state vector S^N, we have

$$\varphi(S^N|p, \mu, \theta, X) = \prod_{j=1}^{N} \varphi(S_j|p, \mu, \theta, X),$$

where $\varphi(S_j|p,\mu,\theta,X)$ is a discrete multinomial distribution explicitly valued by

$$\varphi(S_j = k|p,\mu,\theta,X) = \frac{p_k f_{kj}}{f_j} = \frac{p_k f_{kj}}{\sum_{i=1}^m p_i f_{ij}},\ k = 1,\cdots,m; j = 1,\cdots,N$$

§2.2. Sampling p from $\varphi(p|\mu,\theta,S^N,X)$

From (2), we make out the full conditional posterior density of the group probability vector p depends on merely S^N and X, is independent of parameter vectors μ and θ. Thus , we have

$$\varphi(p|\mu,\theta,S^N,X) \propto \varphi(p)\cdot\varphi(p,\mu,\theta|S^N,X) \propto \varphi(p)\prod_{k=1}^m p_k^{c_k},$$

where $c_k = \sum_{j=1}^N 1_{\{S_j=k\}}$ is the number of times when $S_j = k$. The prior distribution of p is chosen to be a Dirichlet distribution $Di(a_1,\cdots,a_m)$ with hyperparameter $a_1,\cdots,a_m$, hence, we have

$$\begin{aligned}\varphi(p|\mu,\theta,S^N,X) &\propto \frac{\Gamma(\sum_{i=1}^m a_i)}{\prod_{i=1}^m\Gamma(a_i)}\prod_{i=1}^m p_i^{a_i-1}\cdot\prod_{k=1}^m p_k^{c_k}\\ &\sim Di(a_1+c_1,\cdots,a_m+c_m)\end{aligned}$$

§2.3. Sampling μ from $\varphi(\mu|p,\theta,S^N,X)$

From (2), we see the full conditional posterior density of the parameter vector μ depends on only S^N and X, is independent of parameter vectors p and θ. The prior distribution of μ is chosen to be a conjugated multivariate normal distribution $N_m(\mu_0,\Sigma_0)$ with mean hyperparameter vector μ_0 and variance-covariance hyperparameter matrix Σ_0, where $\Sigma_0 > 0$ which denotes variance-covariance hyperparameter matrix Σ_0 is positive definite, so we have

$$\begin{aligned}\varphi(\mu|p,\theta,S^N,X) &\propto exp\{-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\}\cdot\prod_{j=1}^N exp(-\frac{(lnx_j-\mu_{S_j})^2}{2\theta_{S_j}})\\ &\propto exp\{-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\}\cdot\prod_{k=1}^m exp\{-\frac{\sum_{j\in\{S_j=k\}}(\mu_k-lnx_j)^2}{2\theta_k}\}\\ &\propto exp\{-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\}\cdot exp\{-\sum_{k=1}^m\frac{c_k\mu_k^2-2(\sum_{j\in\{S_j=k\}}lnx_j)\mu_k}{2\theta_k}\}\\ &\propto exp\{-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\}\cdot exp\{-\frac{1}{2}\sum_{k=1}^m\lambda_k(\mu_k-\frac{\sum_{j\in\{S_j=k\}}lnx_j}{c_k})^2\}\\ &\propto exp\{-\frac{1}{2}(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\}\cdot exp\{-\frac{1}{2}(\mu-\triangle)^T\Lambda(\mu-\triangle)\}\\ &\propto exp\{-\frac{1}{2}[(\mu-\mu_0)^T\Sigma_0^{-1}(\mu-\mu_0)\}+(\mu-\triangle)^T\Lambda(\mu-\triangle)]\}\\ &\propto exp\{-\frac{1}{2}[\mu-(\mu_0-\Sigma_1U^T)]^T\Sigma_1^{-1}[\mu-(\mu_0-\Sigma_1U^T)]\}\\ &\sim N_m(\mu_0-\Sigma_1U^T,\Sigma_1)\end{aligned}$$

where

$$\lambda_k = \frac{c_k}{\theta_k},\quad \triangle = (\triangle_1,\cdots,\triangle_m)^T$$

$$\triangle_k = \begin{cases} \frac{\sum_{j\in\{S_j=k\}} lnx_j}{c_k} & , c_k \neq 0 \\ 0 & c_k = 0 \end{cases}$$

$$\Lambda = diag(\lambda_1, \cdots, \lambda_m), \quad \Sigma_1^{-1} = \Sigma_0^{-1} + \Lambda, \quad U^T = \Lambda \cdot (\mu_0 - \triangle)$$

§2.4. Sampling θ from $\varphi(\theta|p, \mu, S^N, X)$

By assuming prior independence between the θ_k's, namely, $\varphi(\theta) = \prod_{k=1}^m \varphi(\theta_k)$, we have

$$\varphi(\theta|p, \mu, S^N, X) \propto \varphi(\theta) \cdot \varphi(p, \mu, \theta|S^N, X) \propto \varphi(\theta_1|\mu, \bar{z}^1) \cdot \varphi(\theta_2|\mu, \bar{z}^2) \cdots \varphi(\theta_m|\mu, \bar{z}^m),$$

where $\bar{z}^k = \{x_j|S_j = k\}$, and we obtain

$$\begin{aligned} \varphi(\theta_k|\mu, \bar{x}^k) &\propto \varphi(\theta_k) \prod_{j\in\{S_j=k\}} f_{kj} \\ &\propto \varphi(\theta_k) \prod_{j\in\{S_j=k\}} (\frac{1}{\theta_k})^{1/2} exp\{-\frac{(lnx_j - \mu_k)^2}{2\theta_k}\} \qquad k = 1, \cdots, m \end{aligned}$$

The prior distribution of θ_k is chosen to be a conjugated inverse gamma distribution $I\Gamma(\alpha_k, \beta_k)$ with shape hyperparameter α_k and scale hyperparameter β_k, hence, we have

$$\begin{aligned} \varphi(\theta_k|\mu, \bar{x}^k) &\propto (\frac{1}{\theta_k})^{\alpha_k+1} exp(-\frac{\beta_k}{\theta_k}) \cdot \prod_{j\in\{S_j=k\}} (\frac{1}{\theta_k})^{1/2} exp\{-\frac{(lnx_j - \mu_k)^2}{2\theta_k}\} \\ &\propto (\frac{1}{\theta_k})^{\alpha_k+1} exp(-\frac{\beta_k}{\theta_k}) \cdot (\frac{1}{\theta_k})^{c_k/2} exp\{-\frac{1}{\theta_k} \sum_{j\in\{S_j=k\}} \frac{1}{2}(lnx_j - \mu_k)^2\} \\ &\propto (\frac{1}{\theta_k})^{\alpha_k+c_k/2+1} \cdot exp\{-\frac{1}{\theta_k}[\beta_k + \frac{1}{2} \sum_{j\in\{S_j=k\}} (lnx_j - \mu_k)^2]\} \\ &\sim I\Gamma(\alpha_k + c_k/2, \ \beta_k + \frac{1}{2} \sum_{j\in\{S_j=k\}} (lnx_j - \mu_k)^2) \qquad k = 1, \cdots, m \end{aligned}$$

Since we impose a condition of the independence of the θ_k's, we can sample separately each θ_k from its full conditional posterior density $\varphi(\theta_k|\mu, \bar{x}^k)$.

About the identifying restriction condition $\theta_1 < \cdots < \theta_m$ of mixture component, we can control it by selecting proper prior hyperparameter value. By sampling with repetition from the full conditional densities of parameter vectors (S^N, p, μ, θ), we can obtain a series of samples $(p^1, \mu^1, \theta^1), \cdots, (p^T, \mu^T, \theta^T)$ for parameter vectors (p, μ, θ). In order to ensure sufficient convergence of Markov Chain samples for all parameter, we can get rid of the preceding T_0 samples to make statistical inference by the use of posterior $T - T_0$ samples.

§3. Simulation

In this section, we will carry out some Monte Carlo simulations to test our algorithm effect for estimating unknown parameters of the MLND model (1). We take a two-component mixture(m=2) of log-normal distributions for example. Suppose the real parameters in model (1)

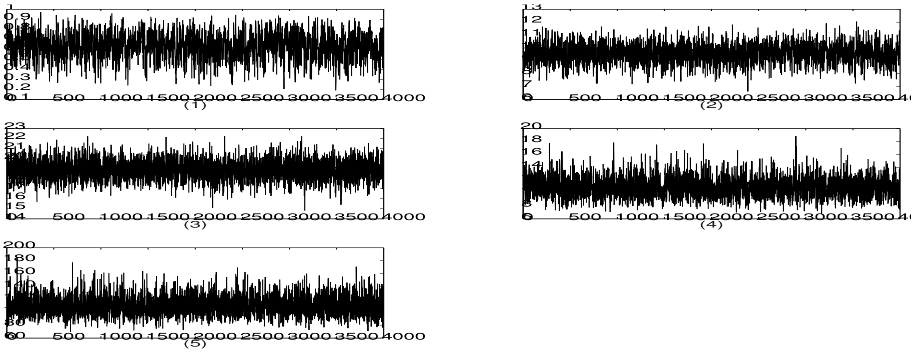

Figure 1: sample paths of the estimated parameters $p_1, \mu_1, \mu_2, \theta_1, \theta_2$ under the sample size n=30

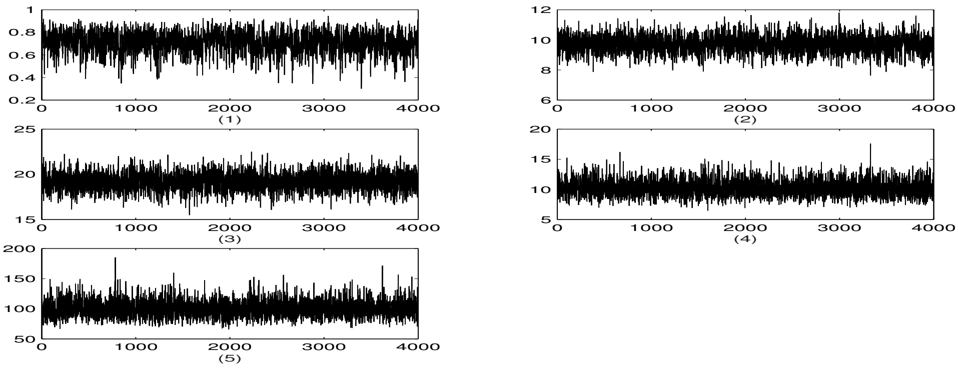

Figure 2: sample paths of the estimated parameters $p_1, \mu_1, \mu_2, \theta_1, \theta_2$ under the sample size n=40

are

$$p = (p_1, 1 - p_1)^T,\ p_1 = 0.7,\ \mu = (10, 20)^T,\ \theta = (10, 100)^T.$$

Hence, there are five real parameters in all. In gibbs sampling, the hyperparameters of Dirichlet distribution, normal distribution, inverse gamma distribution are chosen as following:

$$(a_1, a_2)^T = (1, 1)^T,\ \mu_0 = (9, 19)^T,\ \Sigma_0 = I_{2\times 2},\ (\alpha_1,\ \ \alpha_2)^T = (50, 50)^T,\ (\beta_1, \beta_2)^T = (500, 5000)^T$$

where $I_{2\times 2}$ denotes 2 order identity matrix. We carry out Matlab experiments to circulate gibbs sampling steps 10000 times respectively for all parameters in order for the convergence of Markov chain samples when the sample sizes are $n = 30, 40, 50$. And the front 6000 times samples are got rid of, the posterior 4000 times samples are used to estimate model parameters. The corresponding posterior 4000 sample paths of the five estimated parameters $p_1, \mu_1, \mu_2, \theta_1, \theta_2$ under the sample sizes $n = 30, 40, 50$ see subgraphs (1) (2) (3) (4) (5) respectively in Figure 1, Figure 2 and Figure 3.

If we denote parameter estimation value for unknown parameter vector of the (k)th experiment as following

$$\eta^k = (p_1^k, \mu_1^k, \mu_2^k, \theta_1^k, \theta_2^k).$$

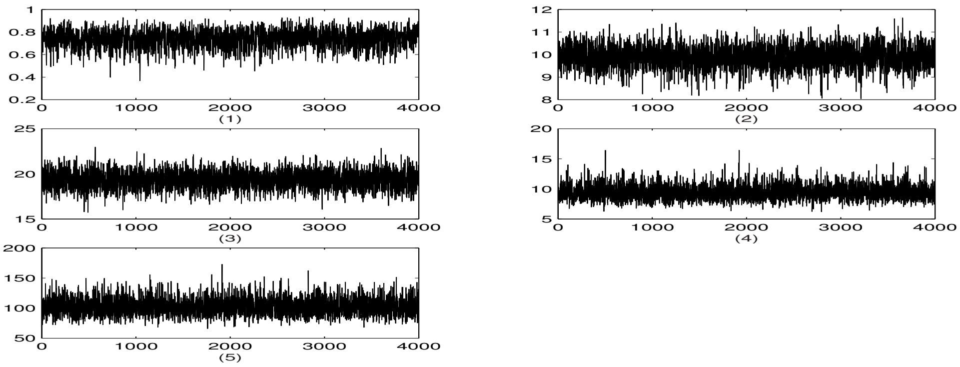

Figure 3: sample paths of the estimated parameters $p_1, \mu_1, \mu_2, \theta_1, \theta_2$ under the sample size n=40

Then, the final means and mean square errors(mse) of the parameters estimation under the squared errors loss function are respectively given by

$$mean_j = (1/4000)\sum_{i=6001}^{10000} \eta_j^i, mse_j = (1/4000)\sum_{i=6001}^{10000} (\eta_j^i - mean_j)^2, j = 1, 2.$$

where η_j is the (j)th coordinate of the unknown parameter vector η. The corresponding computation results see Table 1 and Table 2.

Table 1: Mean of parameters estimation of gibbs sampling algorithm $a \cdot Eb$ **denotes** $a \cdot 10^b$

	$\hat{p_1}$	$\hat{\mu_1}$	$\hat{\mu_2}$	$\hat{\theta_1}$	$\hat{\theta_2}$
$n = 30$	$6.335E-1$	9.564	$1.890E1$	$1.088E1$	$1.102E2$
$n = 40$	$7.111E-1$	$9.723E1$	$1.917E1$	$1.021E1$	$9.979E1$
$n = 50$	$7.393E-1$	$9.883E1$	$1.930E1$	9.458	$1.025E2$

Table 2: Mse of parameters estimation of gibbs sampling algorithm $a \cdot Eb$ **denotes** $a \cdot 10^b$

	$\hat{p_1}$	$\hat{\mu_1}$	$\hat{\mu_2}$	$\hat{\theta_1}$	$\hat{\theta_2}$
$n = 30$	$2.159E-2$	$7.325E-1$	2.185	3.216	$3.342E2$
$n = 40$	$9.375E-3$	$3.979E-1$	1.640	1.840	$1.890E2$
$n = 50$	$7.854E-3$	$2.522E-1$	1.430	1.754	$2.010E2$

From Figure 1, Figure 2, Figure 3 and Table 1, Table 2, we see gibbs sampling algorithm is very effective to the estimation of the unknown parameters of the MLND model (1) and the mean square errors(mse) of most of estimated parameters decrease as n increases.

§4. Conclusion

This paper gives out parameters estimation method of the MLND model (1) by the use of gibbs sampling algorithm in the bayesian framework, and some Monte Carlo simulations are

been carried out to investigate the performance of the estimation technique. The simulation results show gibbs sampling algorithm behaves better than conventional methods to the MLND model (1) even for small sample size under appropriate prior information.

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Some characteristics of k-step state transition probabilities of Markovian process

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Abstract Classical theory of Markov process deals with sequence analysis. Markov process provides a special type of dependence. Few analytic properties of the transition probabilities are analyzed for Markov process and Hidden Markov Model as well. The Gaines algorithm for Hidden Markov Model is found to improve the probability value.

Keywords Markov process, Chapman-Kolmogorov equations, k-step transition probability, Hidden Markov Model, Gaines algorithm.

§1. Introduction

Markov process is a probabilistic description of a series of dependent trials. The Chapman-Kolmogorov equations play an important role in the study of Markov process. Classification of states of a Markov process is based on the probability of its return to its original state. In this paper, some characteristics of k-step transition probability of the system, $p_{ij}(k)$ are analyzed for Markovian process and Hidden Markov Model as well.

§2. Initial state probabilities

Consider a system which may be described at any time as being in any one of the states. It undergoes changes of states at discrete instants of trials from time to time. Let $s_j(n)$ denote that the system is in state s_j after n number of trails. $P[s_j(n)|s_i(n-1)] = p_{ij}$ is the probability that the system is in s_j after n trials provided that it is in s_i after $n-1$ trials. Suppose if the system has m states, say $1, 2, \ldots, m$ then

$$\begin{aligned} P[s_1(n)|s_i(n-1)] + P[s_2(n)|s_i(n-1)] + \cdots + P[s_m(n)|s_i(n-1)] &= 1 \\ p_{i1} + p_{i2} + \cdots + p_{im} &= 1 \\ \sum_{j=1}^{m} p_{ij} &= 1 \text{ for } i = 1, 2, \ldots, m. \end{aligned}$$

The conditional probability that the process will be in state s_j after exactly k trials from a state s_i is called k-step transition probability of the system, denoted by $p_{ij}(k)$. It is easy to

observe that $p_{ij}(1) = p_{ij}$. Also assume that π_i is the probability that the system is in state s_i initially. For simplicity, the states of the given sequence are designated as $0, 1, 2, \ldots, m-1$ and so s_0 is the initial state. If it is the only initial state then $\pi_0 = 1$. Suppose if the system has several initial states then $\sum_i \pi_i = 1$.

Let the system be at s_0 initially and it may be at any one of the states $s_1, s_2, \ldots, s_j$ after j trials, $0 < j < m$. Then

$$\begin{aligned}
&P[s_j(j)|s_j(j-1)\ldots s_1(1)s_0(0)]\\
&= P[s_j(j)|s_{j-1}(j-1)]P[s_{j-1}(j-1)|s_{j-2}(j-2)s_{j-3}(j-3)\ldots s_1(1)s_0(0)]\pi_0\\
&= p_{j-1j}P[s_{j-1}(j-1)|s_{j-2}(j-2)]P[s_{j-2}(j-2)|s_{j-3}(j-3)s_{j-4}(j-4)\ldots s_1(1)s_0(0)]\pi_0\\
&= p_{j-1j}p_{j-2j-1}\cdots p_{12}p_{01}\pi_0
\end{aligned}$$

$$P[s_j(j)|s_j(j-1)\ldots s_1(1)s_0(0)] = \pi_0 \prod_{i=1}^{j} p_{ii-1}$$

The k-step state probabilities have greater significance when the system is at some non-initial state. If the problem is looked in that way the above probability becomes a particular case.

§3. Chapman-Kolmogorov equations

The one step state transition probabilities of the system are represented by means of a matrix as given below.

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{pmatrix}$$

with $\sum_j p_{ij} = 1$ and $p_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, m$.

The matrix $P^k = (p\ p \ldots p)$ k times $= P^{k-1}p$ is called k-step transition probability. In general the equations $P^m = P^{m-k}P^k$, $0 < k < m$ are known as Chapman-Kolmogorov equations. The following theorem paves the way to state that k-step state transition probabilities of a system are a simple case of Chapman-Kolmogorov equations.

Theorem 3.1. Let $P[s_j(n+k)|s_i(n)] = p_{ij}(k)$ is the probability that the system is in state s_j after exactly k more trials from the state s_i. Then

$$P[s_j(n+k)|s_x(n+k-l)] = P[s_j(n+k)|s_x(n+k-l)]P[s_x(n+k-l)|s_i(n)].$$

Proof of Theorem 3.1. Let the system be in the state s_i after n trials. Also assume that the system is in the state s_j after exactly k more trials. In each trial, the system will be in any one or another state of the system. All such states need not be different. In that way, the system will go through some finite number of states before reaching s_j. Let them be

$1, 2, \ldots, k-1$ and k^{th} state be s_j. Then

$$\begin{aligned} P[s_j(n+k)|s_i(n)] &= P[s_j(n+k)|s_{k-1}(n+k-1)s_{k-2}(n+k-2)\ldots s_i(n)] \\ &= P[s_j(n+k)|s_{k-1}(n+k-1)]P[s_{k-1}(n+k-1)|s_{k-2}(n+k-2)\ldots \\ &\quad s_1(n+1)s_i(n)] \\ &= p_{k-1j}P[s_{k-1}(n+k-1)|s_{k-2}(n+k-2)]P[s_{k-2}(n+k-2)|s_{k-3}(n+k-3) \\ &\quad s_{k-4}(n+k-4)\ldots s_1(n+1)s_i(n)] \\ &= p_{k-1\ j}p_{k-2\ k-1}\ldots p_{i1}\pi_i \end{aligned}$$

where π_i is the probability that the system is in state s_i after n trials.

$$\begin{aligned} i.e., P[s_j(n+k)|s_i(n)] &= p_{k-1j}p_{k-2k-1}\ldots p_{i1}\pi_i \\ &= p_{k-1j}p_{k-2k-1}\ldots p_{x-1x}\ldots p_{i1}\pi_i \\ &= p_{k-1j}p_{k-2k-1}\ldots p_{xx+1}p_{x-1x}\ldots p_{i1}\pi_i \end{aligned}$$

So let $P[s_x(n+k-l)|s_i(n)] = p_{x-1x}\ldots p_{i1}\pi_i$.
Now $P[s_j(n+k)|s_x(n+k-l)] = p_{k-1j}p_{k-2k-1}\ldots p_{xx+1}\pi_x$, where π_x is the probability that the system is in x^{th} state after $n+k-l$ trials and is given by $\pi_x = p_{x-1x}p_{x-2x-1}\ldots p_{i1}\pi_i$. Hence $P[s_j(n+k)|s_x(n+k-l)] = P[s_j(n+k)|s_x(n+k-l)]P[s_x(n+k-l)|s_i(n)]$.

Definition 3.2. A system X is said to be a discrete state discrete transitive Markov process if $P[s_j(n)|s_a(n-1)s_b(n-1)\ldots] = P[s_j(n)|s_a(n-1)]$. This condition is called Markov condition.

Theorem 3.3. In a discrete state discrete transition Markov process

$$p_{ij}(k) = \sum_{x=1}^{m} p_{ix}(k-l)p_{xj}(l), \quad k = 1, 2, 3, \ldots;\ 0 \leq l \leq k;\ 1 \leq i, j \leq m.$$

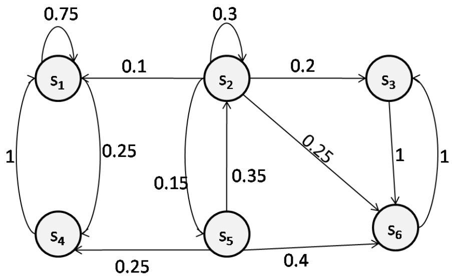
Proof of Theorem 3.3.

$$\begin{aligned} p_{ij}(k) &= P[s_j(n+k)|s_i(n)] \\ &= \sum_{x=1}^{m} P[s_j(n+k)s_x(n+k-l)|s_i(n)] \\ &= \sum_{x=1}^{m} P[s_x(n+k-l)|s_i(n)]P[s_j(n+k)|s_x(n+k-l)s_i(n)] \\ &= \sum_{x=1}^{m} P[s_x(n+k-l)|s_i(n)]P[s_j(n+l)|s_x(n)] \\ &= \sum_{x=1}^{m} p_{ix}(k-l)p_{xj}(l). \end{aligned}$$

This is a simple case of Chapman-Kolmogorov equations.

§4. Limiting values of k-step state transition probabilities

Consider the Markov process given in the following figure.



Applying Theorem 3.3 for the above process table 1 gives the limiting values.

Table 1 informs, for instance, given that the process is in state s_1 at any time, the conditional probability that the process will be in state s_4 exactly after 5 trails is 0.201. It appears that the k-step transition probabilities $p_{ij}(k)$ reach an accumulation point or converge as k increases. As $k \to \infty$, the k-step state transition probabilities $p_{ij}(k)$ do not depend on k or i.

Theorem 4.1. Let $P[s_i(0)]$ be the probability that the process is in state s_i just before the first trail. Then

$$P[s_j(k)] = \sum_{i=1}^{m} P[s_i(0)]p_{ij}(k).$$

Proof of Theorem 4.1. The quantities $P[s_i(0)]$ are known as the initial probabilities for the process and $\sum_{x=1}^{m} P[s_i(0)] = 1$. Consider $P[s_j(k)] = P[X_{n+k} = s_j]$ after $(n+k)$ trails.

Initially the process can be in any one of the states s_i, $i = 1$ to m and after $(n+k)$ trails the process may be in any one of the states s_j, $j = 1$ to m.

$$\begin{aligned} i.e., P[X_{n+k} = s_j] &= P[X_0 = s_1]P[X_{n+k} = s_j|X_n = s_i] + P[X_0 = s_2]P[X_{n+k} = s_j|X_n = s_i] \ldots \\ &\quad + P[X_0 = s_m]P[X_{n+k} = s_j|X_n = s_i] \\ &= [P[X_0 = s_1] + P[X_0 = s_2] + \cdots + P[X_0 = s_m]]P[X_{n+k} = s_j|X_n = s_i] \\ &= \sum_{i=1}^{m} P[X_0 = s_i]P[X_{n+k} = s_j|X_n = s_i] \\ &= \sum_{i=1}^{m} P[s_i(0)]p_{ij}(k). \end{aligned}$$

As mentioned earlier when $k \to \infty$ the probability $p_{ij}(k)$ depends neither on k nor on i. It can be concluded from the above equation that $P[s_j(k)]$ approaches a constant as $k \to \infty$ and this constant is independent of the initial conditions. Hence $\lim_{k\to\infty} P[s_j(k)] = P_j$, $j = 1, 2, \ldots, m$ exist and converge to a probability value.

§5. State classification and convergence

Definition 5.1. State s_i is called transient if there exists a state s_j and an integer l such that $p_{ij}(l) \neq 0$ and $p_{ji}(k) = 0$ for $k = 0, 1, 2, \cdots$.

In other words s_i is a transient state if there exists any state to which the system (in some number of trials) can reach from s_i but it can never return to s_i.

<table>
<tr><th>$p_{ij}(k)$</th><th>$k=1$</th><th>$k=2$</th><th>$k=3$</th><th>$k=4$</th><th>$k=5$</th><th>$k=6$</th><th>$k=7$</th><th>$k=8$</th><th>$k=9$</th></tr>
<tr><td>$p_{11}(k)$</td><td>0.75</td><td>0.813</td><td>0.798</td><td>0.802</td><td>0.801</td><td>0.8</td><td>0.8</td><td>0.8</td><td>0.8</td></tr>
<tr><td>$p_{14}(k)$</td><td>0.25</td><td>0.182</td><td>0.202</td><td>0.198</td><td>0.199</td><td>0.201</td><td>0.2</td><td>0.2</td><td>0.2</td></tr>
<tr><td>$p_{21}(k)$</td><td>0.10</td><td>0.105</td><td>0.156</td><td>0.16</td><td>0.168</td><td>0.17</td><td>0.171</td><td>0.171</td><td>0.171</td></tr>
<tr><td>$p_{22}(k)$</td><td>0.30</td><td>0.146</td><td>0.06</td><td>0.026</td><td>0.0113</td><td>0.005</td><td>0.002</td><td>0.002</td><td>0.002</td></tr>
<tr><td>$p_{23}(k)$</td><td>0.2</td><td>0.31</td><td>0.364</td><td>0.438</td><td>0.393</td><td>0.451</td><td>0.398</td><td>0.453</td><td>0.4</td></tr>
<tr><td>$p_{24}(k)$</td><td>0</td><td>0.063</td><td>0.037</td><td>0.045</td><td>0.043</td><td>0.043</td><td>0.043</td><td>0.043</td><td>0.043</td></tr>
<tr><td>$p_{25}(k)$</td><td>0.15</td><td>0.045</td><td>0.22</td><td>0.01</td><td>0.004</td><td>0.002</td><td>0.001</td><td>0.001</td><td>0.001</td></tr>
<tr><td>$p_{26}(k)$</td><td>0.25</td><td>0.335</td><td>0.365</td><td>0.388</td><td>0.449</td><td>0.397</td><td>0.453</td><td>0.4</td><td>0.4</td></tr>
<tr><td>$p_{31}(k)$</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td></tr>
<tr><td>$p_{32}(k)$</td></tr>
<tr><td>$p_{34}(k)$</td></tr>
<tr><td>$p_{35}(k)$</td></tr>
<tr><td>$p_{33}(k)$</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>$p_{36}(k)$</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>$p_{41}(k)$</td><td>1</td><td>0.75</td><td>0.813</td><td>0.797</td><td>0.801</td><td>0.8</td><td>0.8</td><td>0.8</td><td>0.8</td></tr>
<tr><td>$p_{42}(k)$</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td></tr>
<tr><td>$p_{43}(k)$</td></tr>
<tr><td>$p_{45}(k)$</td></tr>
<tr><td>$p_{46}(k)$</td></tr>
<tr><td>$p_{44}(k)$</td><td>0</td><td>0.25</td><td>0.187</td><td>0.203</td><td>0.199</td><td>0.2</td><td>0.2</td><td>0.2</td><td>0.2</td></tr>
<tr><td>$p_{51}(k)$</td><td>0</td><td>0.285</td><td>0.2243</td><td>0.258</td><td>0.256</td><td>0.261</td><td>0.262</td><td>0.264</td><td>0.264</td></tr>
<tr><td>$p_{52}(k)$</td><td>0.35</td><td>0.105</td><td>0.05</td><td>0.0206</td><td>0.009</td><td>0.004</td><td>0.0017</td><td>0.001</td><td>0.001</td></tr>
<tr><td>$p_{53}(k)$</td><td>0</td><td>0.47</td><td>0.109</td><td>0.528</td><td>0.132</td><td>0.538</td><td>0.137</td><td>0.54</td><td>0.54</td></tr>
<tr><td>$p_{54}(k)$</td><td>0.25</td><td>0</td><td>0.085</td><td>0.06</td><td>0.067</td><td>0.65</td><td>0.066</td><td>0.066</td><td>0.066</td></tr>
<tr><td>$p_{55}(k)$</td><td>0</td><td>0.053</td><td>0.016</td><td>0.008</td><td>0.0031</td><td>0.0014</td><td>0.001</td><td>0.003</td><td>0.003</td></tr>
<tr><td>$p_{56}(k)$</td><td>0.40</td><td>0.088</td><td>0.518</td><td>0.1279</td><td>0.536</td><td>0.136</td><td>0.54</td><td>0.138</td><td>0.138</td></tr>
<tr><td>$p_{61}(k)$</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td><td rowspan="4">0</td></tr>
<tr><td>$p_{62}(k)$</td></tr>
<tr><td>$p_{64}(k)$</td></tr>
<tr><td>$p_{65}(k)$</td></tr>
<tr><td>$p_{63}(k)$</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
<tr><td>$p_{66}(k)$</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>1</td></tr>
</table>

Definition 5.2. A state s_i is called recurrent if the process eventually returns to s_i in

some number of transitions.

$$P[X_{n+k} = s_i | X_n = s_i] = 1, \ n \geq 1.$$

Every state must be either recurrent or transient. In the above example (Figure 1) states s_2 and s_4 are the only transient states; remaining states s_1, s_3, s_5 and s_6 are recurrent states. If any two states are recurrent, it is not necessary that the process can never get from one of the states to the other. It could be noted that in the example $p_{13}(k) = 0$ and $p_{31}(k) = 0$, states 1 and 3 being recurrent. The results are same for states 4 and 6.

Definition 5.3. A recurrent state s_i is called periodic if there exist an integer d with $d \geq 2$, such that $p_{ii}(k) = 0$ for all values of k other than $d, 2d, 3d, \ldots$.

In the above example, recurrent states s_3 and s_6 are the only periodic states. A set of recurrent states with the property that the system can eventually get from any member of state to any other state which is also a member of the chain. All possible members of each such one are included in the chain.

In a Markov process with a finite number of states, the states are either transient or recurrent. Those recurrent states which form a single chain do not contain periodic states. The k-step transition probabilities $p_{ij}(k)$ of the recurrent states of a single chain become independent of i and k, as $k \to \infty$. For a Markov process with a finite number of states, whose recurrent states from a single chain, which contains no periodic states then

$$\lim_{k\to\infty} p_{ij}(k) = P_j, \qquad \sum_{j=1}^{m} P_j = 1$$

where P_j depends neither on k nor on i. This P_j reaches a converging point and will be in states s_j after many trails, whatever may be the initial condition.

In the example mentioned above there are two single chains of recurrent states. One chain is of states s_1 and s_4 and the other chain includes s_3 and s_6. It could be observed that $\lim\limits_{k\to\infty} p_{11}(k) = 0.8$, $\lim\limits_{k\to\infty} p_{14}(k) = 0.2$ with $\lim\limits_{k\to\infty} p_{12}(k) = \lim\limits_{k\to\infty} p_{13}(k) = \lim\limits_{k\to\infty} p_{15}(k) = \lim\limits_{k\to\infty} p_{16}(k) = 0$ proving $\sum\limits_{j=1}^{6} P_j = 1$ and similarly it proves the same for s_3 and s_6.

From the table it could be observed that the probabilities of transient states s_2 and s_5 after many trails converge to a constant value. i.e., $\lim_{k\to\infty} p_{2j}(k) = P_j, \lim_{k\to\infty} p_{5j}(k) = P_j, j =$ 1 to 6.

§6. Limiting probabilities for Hidden Markov Model

Definition 6.1. An Hidden Markov Model (HMM) is mathematically equal to a stochastic finite automaton defined by a 5-tuple $A = (Q, \Sigma, \Delta, \pi, O)$ where $Q = \{s_0, s_1, s_2, s_3, \ldots, s_m\}$ is finite set of states, Σ is an alphabet of output symbols, $\Delta = \{p_{ij}/1 \leq i, j \leq m\}$ is a state transition probability distribution and $\pi = \{\pi_i/1 \leq i \leq m\}$ is an initial state distribution, O is the set $\{e_j(x)/1 \leq j \leq n\}$ of output symbol probabilities such that

$$\sum_{j=1}^{m} p_{ij} = 1, \quad \sum_{x\in\Sigma} e_j(x) = 1 \ \text{ and } \ \sum_{i=1}^{m} \pi_i = 1$$

In general, given an HMM there are three basic problems:

(i) calculating the probability of the string generated by HMM.

(ii) finding the most probable path.

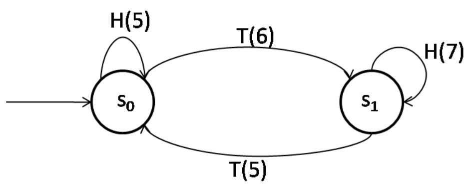
(iii) estimating the parameters to maximize the probability.

The Probabilistic Finite State Automata (PFSA) constructed here [2] for a given data are restricted to have, from each state at most one transition with a given output symbol. Hence there exists exactly one path from initial state.

Definition 6.2. The probability of the sequence is the product of the probabilities of initial state, the output symbols, and the state transitions which generate the given string [7].

i.e., if $I = x_1 \ldots x_m$ is the data and A is a two state automaton generating I, then the probability $P(I/A) = \pi_1 e_1(x_1) a_{12} e_2(x_2) \ldots e_{m-1}(x_{m-1}) a_{m-1,m} e_m(x_m)$. Consider a sequence like HTHHTHHTTTHTTTHHTHHHHTT ....while tossing a coin.

Let v be the number of symbols in the data with probabilities e_i, $i = 1, 2, 3, \ldots, v$, and $\sum e_i = 1$. If a state is visited 't' times, the probability e_i is estimated approximately as $(n_i + 1/2)/(t + v/2)$, n_i is the number of times the i^{th} symbol is produced from the particular state. The derivations of this Gaines algorithm can be seen in Wallace and Boulton [6].



The HMM that suits the given data is defined as $A = (Q, \Sigma, \Delta, \pi, O)$ where $Q = \{s_0, s_1\}$, $\Sigma = \{H, T\}$, $\Delta = \{p_{11} = 0.45, p_{12} = 0.55, p_{21} = 0.42, p_{22} = 0.58\}$, $O = \{e_1(H) = 0.46, e_1(T) = 0.54, e_2(H) = 0.58, e_2(T) = 0.42\}$, $\pi = \{\pi_1 = 1, \pi_2 = 0\}$. The probability of I that is generated by A is given by $P(I/A) = \pi_1 e_1(H)^5 p_{11}^5 e_1(T)^6 p_{12}^6 e_2(H)^7 p_{22}^7 e_2(T)^5 p_{21}^5 = 2.172 \times 10^{-14}$.

The above HMM is a single chain of recurrent states as in above Markov process with finite number of states [1]. After many trials it can be proved that $\sum_{j=1}^{2} P_j = 1$ for states s_0 and s_1.

While applying Theorem 3.3 for the transition probabilities of the above HMM, the following are the results observed.

$p_{ij}(k)$	$k = 1$	$k = 2$	$K = 3$	$K = 4$	$k = 5$	$k = 6$
$p_{11}(k)$	0.45	0.4335	0.433	0.433	0.433	0.433
$p_{12}(k)$	0.55	0.5665	0.567	0.567	0.567	0.567
$p_{21}(k)$	0.4326	0.433	0.433	0.433	0.433	0.433
$p_{22}(k)$	0.5674	0.567	0.567	0.567	0.567	0.567

The probability value $P(I/A)$ using these limiting transition probabilities is 2.1372×10^{-14}. It could be seen that the probability value of a HMM is greater through Gaines algorithm than limiting the transition probabilities.

The Baum-Welch algorithm [3] is an expectation maximization algorithm and it is applied to calculate the expected number of times each transition or emission is used to adjust the parameters to maximize the likelihood of expected values. In this attempt the number of

transitions and their probabilities are kept fixed for the HMM and Baum - Welch algorithm is used only to improve the emission probabilities. With the transition probabilities $p_{11} = 0.45$, $p_{12} = 0.55$, $p_{21} = 0.42$, $p_{22} = 0.58$, using Baum - Welch algorithm it is found that $e_1(H) = 0.4553$; $e_1(T) = 0.5447$; $e_2(H) = 0.5727$; $e_2(T) = 0.4274$. The new probability $P(I/A) = 2.1680 \times 10^{-14}$. This probability value is less than 2.172×10^{-14}. Thus the emission probabilities for the HMM, based on the new algorithm improves and estimates the highest possible probability of a given sequence.

Conclusion

The Chapman-Kolmogorov equations are used to limit the transition probabilities in a Markov process. The experimental results show that an HMM optimized through Gaines algorithm improves the probability value directly than limiting the transition probabilities. The emission probabilities of new HMM algorithm contributes the maximum in obtaining the improved probability value.

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Smarandache's Cevians theorem (I)

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Abstract We present the Smarandache's Cevians theorem in the geometry of the triangle.

Keywords Smarandache's Cevians theorem, geometry, triangle.

§1. The main result

Smarandache's Cevians Theorem (I)

In a triangle $\triangle ABC$, let's consider the Cevians AA', BB' and CC' that intersect in P. Then:

$$E(P) = \frac{\| PA \|}{\| PA' \|} + \frac{\| PB \|}{\| PB' \|} + \frac{\| PC \|}{\| PC' \|} \geq 6,$$

and

$$F(P) = \frac{\| PA \|}{\| PA' \|} \cdot \frac{\| PB \|}{\| PB' \|} \cdot \frac{\| PC \|}{\| PC' \|} \geq 8,$$

where $A' \in [BC], B' \in [CA], C' \in [AB]$.

Proof. We'll apply the theorem of Van Aubel three times for the triangle $\triangle ABC$, and it results:

$$\frac{\| PA \|}{\| PA' \|} = \frac{\| AC' \|}{\| C'B \|} + \frac{\| AB' \|}{\| B'C \|}.$$

$$\frac{\| PB \|}{\| PB' \|} = \frac{\| BA' \|}{\| A'C \|} + \frac{\| BC' \|}{\| C'A \|}.$$

$$\frac{\| PC \|}{\| PC' \|} = \frac{\| CA' \|}{\| A'B \|} + \frac{\| CB' \|}{\| B'A \|}.$$

If we add these three relations and we use the notation

$$\frac{\| AC' \|}{\| C'B \|} = x > 0, \frac{\| AB' \|}{\| B'C \|} = y > 0, \frac{\| BA' \|}{\| A'C \|} = z > 0,$$

then we obtain:

$$E(P) = (x + \frac{1}{y}) + (x + \frac{1}{y}) + (z + \frac{1}{z}) \geq 2 + 2 + 2 = 6.$$

The minimum value will be obtained when $x = y = z = 1$, therefore when P will be the gravitation center of the triangle.

When we multiply the three relations we obtain

$$F(P) = (x + \frac{1}{y}) \cdot (x + \frac{1}{y}) \cdot (z + \frac{1}{z}) \geq 8.$$

§2. Open problems related to Smarandache's Cevians theorem (I)

1. Instead of a triangle we may consider a polygon $A_1A_2\ldots A_n$ and the lines A_1A_1', A_2A_2', $\ldots, A_nA_n'$ that intersect in a point P.

Calculate the minimum value of the expressions:

$$E(P) = \frac{\parallel PA_1 \parallel}{\parallel PA_1' \parallel} + \frac{\parallel PA_2 \parallel}{\parallel PA_2' \parallel} + \ldots + \frac{\parallel PA_n \parallel}{\parallel PA_n' \parallel},$$

$$F(P) = \frac{\parallel PA_1 \parallel}{\parallel PA_1' \parallel} \cdot \frac{\parallel PA_2 \parallel}{\parallel PA_2' \parallel} \cdot \ldots \cdot \frac{\parallel PA_n \parallel}{\parallel PA_n' \parallel}.$$

2. Then let's generalize the problem in the $3D$ space, and consider the polyhedron $A_1A_2\ldots A_n$ and the lines $A_1A_1', A_2A_2', \ldots, A_nA_n'$ that intersect in a point P. Similarly, calculate the minimum of the expressions $E(P)$ and $F(P)$.

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On quotient binary algebras

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Abstract In this paper, we introduce binary algebra and quotient binary algebra. We show the relation between ideals and congruences on binary algebra.

Keywords Ideal, congruence, binary algebras.

§1. Introduction

By an algebra $X = (X, *, 0)$, we mean a non-empty set X together with a binary operation $*$ and a some distinguished 0.

Y. L. Liu et al. [4] studied an algebraic structure called a *BCI-algebra* which is an algebra $(X, *, 0)$ with a binary operation $*$ such that for all $x, y, z \in X$, satisfies the following properties:

(1) $((x * y) * (x * z)) * (z * y) = 0$;

(2) $(x * (x * y)) * y = 0$;

(3) $x * x = 0$;

(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In 2003, E. H. Roh et al. [3] introduced an algebra $(X; *, \leq, 0)$ with a binary operation $*$ and a nullary operation 0. Moreover, a binary relation $\leq$ on X is called *difference algebra* if it satisfies for all $x, y, z \in X$:

(D1) $(X, \leq)$ is a poset;

(D2) $x \leq y$ implies $x * z \leq y * z$;

(D3) $(x * y) * z \leq (x * z) * y$;

(D4) $0 \leq x * x$;

(D5) $x \leq y$ if and only if $x * y = 0$.

In this paper, we first introduce an algebraic structure called a *binary algebra* and study the relation between ideals and congruences. Finally, we define *quotient binary algebra* and study its properties.

§2. Binary algebras

We define a binary algebras as follows:

Definition 2.1. An algebra $(X; *, 0)$ with a binary operation $*$ and a nullary operation 0. Then X is called *binary algebra* if it satisfies for all $x, y, z \in X$:

(B-1) $((x * y) * (x * z)) * (z * y) = 0$;

(B-2) $x * x = 0$;

(B-3) $x * 0 = x$;

(B-4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

(B-5) $(x * y) * z = (x * z) * y$;

It is easy to show that the following properties are true for a binary algebra. For all x, y, z in X :

(1) $(x * (x * y)) * y = 0$;

(2) $((x * z) * (y * z)) * (x * y) = 0$;

(3) $x * y = 0$ imply $(x * z) * (y * z) = 0$ and $(z * y) * (z * x) = 0$

We now show the relationship between a binary algebra and BCK/BCC-algebras. If X is a binary algebra and it satisfies $0 * x = 0$ for all $x \in X$, then X is a BCK-algebra. Conversely, if X is a BCK-algebra which satisfies $(x * y) * z = (x * z) * y$ and $x * 0 = x$ for all $x, y, z \in X$, then X is a binary algebra. Similarly, if X is a binary algebra and it satisfies $0 * x = 0$ for all $x \in X$, then X is a BCC-algebra. Conversely, if X is a BCC-algebra and fulfils $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$, then X is a binary algebra. In [2], any BCK-algebra is a BCC-algebra. In [1], a BCC-algebras is a BCK-algebras iff it satisfies $(x * y) * z = (x * z) * y$.

Examples 2.2. Let $\mathbb{R}$ be the set of all real number, the unit of which is 0, the operation $-$ is the inverse of the operation $+$, that is $x - y = x + (-y)$ for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}; -, 0)$ is a binary algebra.

Definition 2.3. A non-empty subset A of a binary algebra X is called a *closed* of X on condition that $x * y \in A$ whenever $x, y \in A$.

Definition 2.4. A non-empty subset A of a binary algebra X is called an *ideal* of X if it satisfies the following conditions:

(I-1) $0 \in A$

(I-2) for any $x, y \in X$, $x * y \in A$ and $y \in A$ imply $x \in A$.

Examples 2.5. Let $X = \{0, 1, 2, 3\}$ and let $*$ be defined by the table

$*$	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Thus, it can be easily shown that X is a binary algebra. Note that $I = \{0, 1\}$ and $J = \{0, 3\}$ are closed ideals of X.

Examples 2.6. $(\mathbb{R}; -, 0)$ is a binary algebra, where $\mathbb{R}$ is a set of real numbers and binary operation $-$ is defined by $x - y = x + (-y)$ with an addition identity 0. It can be easily checked that $\mathbb{R}_0^+$ and $\mathbb{R}_0^-$ are ideal of $(\mathbb{R}; -, 0)$. But $\mathbb{R}_0^+, \mathbb{R}_0^-$ are not closed of $\mathbb{R}$.

Lemma 2.7. Let A be a closed of binary algebra X. Then A is an ideal of X if and only if $x \in A$ and $y * z \notin A$ imply $(y * x) * z \notin A$ for all $x, y, z \in X$.

Proof. Let A be an ideal of X and let $x \in A$ whereas $y*z \notin A$. Suppose that $(y*x)*z \in A$. By B-5, we see that $(y*z)*x \in A$. Since A is an ideal of X and $x \in A$, $y*z \in A$, a contradiction. So $(y * x) * z \notin A$.

Conversely, assume that $x \in A$ and $y * z \notin A$ imply $(y * x) * z \notin A$ for all $x, y, z \in X$. Since A is a closed of X, there is $x \in A$ which $0 = x * x \in A$. That is, $0 \in A$. Let $y * x \in A$ and $x \in A$. Assume that $y \notin A$. We have that $y * 0 = y \notin A$. It follows that $(y * x) * 0 \notin A$. Hence $y * x \notin A$, a contradiction. This completes the proof.

Corollary 2.8. Let A be a closed of binary algebra X. Then A is an ideal of X if and only if $x \in A$ and $y \notin A$ imply $y * x \notin A$ for all $x, y \in X$.

Lemma 2.9. Let A be a closed of binary algebra X. Then A is an ideal of X if and only if $(y * x) * z \in A$ and $y * z \notin A$ imply $x \notin A$ for all $x, y, z \in X$.

Proof. Let A be an ideal of X and let $(y * x) * z \in A$, $y * z \notin A$. Suppose that $x \in A$. By B-5, we have $(y * z) * x \in A$. Since A is an ideal of X, $y * z \in A$, a contradiction, this shows that $x \notin A$.

Conversely, assume that $(y * x) * z \in A$ and $y * z \notin A$ imply $x \notin A$ for all $x, y, z \in X$. Since A is a closed of X, there is $x \in A$ which $0 = x * x \in A$. Then $0 \in A$. Let $y * x \in A$, $x \in A$ and suppose that $y \notin A$. By B-3, $(y * x) * 0 \in A$ and $y * 0 \notin A$. Hence $x \notin A$, a contradiction. This proves that A is an ideal of X.

Corollary 2.10. Let A be a closed of binary algebra X. Then A is an ideal of X if and only if $y * x \in A$ and $y \notin A$ imply $x \notin A$ for all $x, y, z \in X$.

This Corollary gives some properties of ideal of binary algebra.

Lemma 2.11. If A is an ideal of binary algebra X and B is an ideal of A, then B is an ideal of X.

Proof. Since B is an ideal of A, then $0 \in B$. Let $x, y \in B$ such that $x * y \in B$ and $y \in B$. We see that $x * y \in A$ and $y \in A$. By assumption, A is an ideal of X, it follows that $x \in B$. Therefore, B is an ideal of X.

Theorem 2.12. Let $\{J_i : i \in \mathbb{N}\}$ be a family of ideals of a binary algebra X where $J_n \subseteq J_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X.

Proof. Let $\{J_i : i \in \mathbb{N}\}$ be a family of ideals of X. It can be proved easily that $\bigcup_{n=1}^{\infty} J_n \subseteq X$. Since J_i is an ideal of X for all i, so $0 \in \bigcup_{n=1}^{\infty} J_n$. Let $x * y \in \bigcup_{n=1}^{\infty} J_n$ and $y \in \bigcup_{n=1}^{\infty} J_n$. It follows that $x * y \in J_j$ for some $j \in \mathbb{N}$ and $y \in J_k$ for some $k \in \mathbb{N}$. WLOG, we assume $J_j \subseteq J_k$. Hence $x * y \in J_k$ and $y \in J_k$. By assumption, J_k is an ideal of X, it follows that $x \in J_k$. Therefore, $x \in \bigcup_{n=1}^{\infty} J_n$, proving that $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X.

Theorem 2.13. Let $\{J_i : i \in \mathbb{N}\}$ be a family of closed ideals of a binary algebra X where $J_n \subseteq J_{n+1}$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} J_n$ is a closed ideal of X.

Proof. Let $\{J_i : i \in \mathbb{N}\}$ be a family of closed ideals of X. By Theorem 2.12, $\bigcup_{n=1}^{\infty} J_n$ is an ideal of X. We will show that $\bigcup_{n=1}^{\infty} J_n$ is a closed of X. Let $x, y \in \bigcup_{n=1}^{\infty} J_n$. It follows that $x \in J_j$ for some $j \in \mathbb{N}$ and $y \in J_k$ for some $k \in \mathbb{N}$. WLOG, we assume that $j \leq k$, we obtain $J_j \subseteq J_k$. That is, $x \in J_k$ and $x \in J_k$. Since J_k is a closed of X, we get $x * y \in J_k \subseteq \bigcup_{n=1}^{\infty} J_n$. This proves that $\bigcup_{n=1}^{\infty} J_n$ is a closed ideal of X.

Theorem 2.14. Let $\{I_j : j \in J\}$ be a family of ideals of a binary algebra X. Then $\bigcap_{j \in J} I_j$ is an ideal of X.

Proof. Let $\{I_j : j \in J\}$ be a family of ideals of X. It is obvious that $\bigcap_{j \in J} I_j \subseteq X$. Since $0 \in I_j$ for all $j \in J$, it follows that $0 \in \bigcap_{j \in J} I_j$. Let $x * y \in \bigcap_{j \in J} I_j$ and $y \in \bigcap_{j \in J} I_j$. We get that $x * y \in I_j$ and $y \in I_j$ for all $j \in J$, then $x \in \bigcap_{j \in J} I_j$ for all $j \in J$ because I_j is an ideal of X. Hence $x \in \bigcap_{j \in J} I_j$, proving our Theorem.

Theorem 2.15. Let $\{I_j : j \in J\}$ be a family of closed ideals of a binary algebra X. Then $\bigcap_{j \in J} I_j$ is a closed ideal of X.

Proof. Let $\{I_j : j \in J\}$ be a family of closed ideals of X. By Theorem 2.14, $\bigcap_{j \in J} I_j$ is an ideal of X. We will show that $\bigcap_{j \in J} I_j$ is a closed of X. Let $x, y \in \bigcap_{j \in J} I_j$. It follows that $x, y \in I_j$ for all $j \in J$. Since I_j is a closed of X and $x * y \in I_j$ for all $j \in J$, then $x * y \in \bigcap_{j \in J} I_j$. This show that $\bigcap_{j \in J} I_j$ is a closed ideal of X.

§3. Quotient binary algebras

In this section, we describe congruence on binary algebras.

Definition 3.1. Let I be an ideal of a binary algebra X. Define a relation $\sim$ on X by

$$x \sim y \text{ iff } x * y \in I \text{ and } y * x \in I.$$

Theorem 3.2. If I is an ideal of binary algebra X, then the relation $\sim$ is an equivalence relation on X.

Proof. Let I be an ideal of X and $x, y, z \in X$. By B-2, $x * x = 0$ and assumption, $x * x \in I$. That is, $x \sim x$. Hence $\sim$ is reflexive. Moreover, suppose that $x \sim y$. Then $x * y \in I$ and $y * x \in I$.

we see that $x \sim y$, so $\sim$ is symmetric. Finally, let $x \sim y$ and $y \sim z$. Then $x*y, y*x, y*z, z*y \in I$ and $((x*z)*(y*z))*(x*y) = ((x*z)*(x*y))*(y*z) = 0 \in I$. We have that $y*z \in I$ and $z*y \in I$, so $x*z \in I$. Similarly, $z*x \in I$. Thus $\sim$ is transitive. Therefore, $\sim$ is an equivalence relation.

Lemma 3.3. Let I be an ideal of binary algebra X and $x, y, u, v \in X$. If $x \sim y$ and $u \sim v$, then $x*u \sim y*v$.

Proof. If $x \sim y$ and $u \sim v$ for $x, y, u, v \in X$, then $x*y, y*x, u*v, v*u \in I$ and by B-1, we see that $((x*u)*(x*v))*(v*u) = 0$ and $((x*v)*(x*u))*(u*v) = 0$. By assumption and I is an ideal of X, these imply that $(x*u)*(x*v) \in I$ and $(x*v)*(x*u) \in I$. This shows that $x*u \sim x*v$. On the other hand, we have that $((x*v)*(y*v))*(x*y) = 0$ and $((y*v)*(x*v))*(y*x) = 0$. By assumption and I is an ideal of X, these imply that $(x*v)*(y*v) \in I$ and $(y*v)*(x*v) \in I$. Thus $x*v \sim y*v$. Since $\sim$ is transitive, so $x*u \sim y*v$.

Corollary 3.4. If I is an ideal of binary algebra X, then the relation $\sim$ is a congruence relation on X.

Proof. By Theorem 3.2 and Lemma 3.3.

Definition 3.5. Let I be an ideal of a binary algebra X. Given $x \in X$, the equivalence class $[x]_I$ of x is defined as the set of all element of X that are equivalent to x, that is,

$$[x]_I = \{y \in X : x \sim y\}.$$

We define the set $X/I = \{[x]_I : x \in X\}$ and a binary operation $\circ$ on X/I by

$$[x]_I \circ [y]_I = [x*y]_I$$

Theorem 3.6. If I is an ideal of binary algebra X with $X/I = \{[x]_I : x \in X\}$ where a binary operation $\circ$ on a set X/I is defined by $[x]_I \circ [y]_I = [x*y]_I$, then the binary operation $\circ$ is a mapping from $X/I \times X/I$ to X/I.

Proof. Let $[x_1]_I, [x_2]_I, [y_1]_I, [y_2]_I \in X/I$ such that $[x_1]_I = [x_2]_I$ and $[y_1]_I = [y_2]_I$. It follows that $x_1 \sim x_2$ and $y_1 \sim y_2$. By Lemma 3.3, $x_1 * y_1 \sim x_2 * y_2$, proving that $[x_1 * y_1]_I = [x_2 * y_2]_I$.

Theorem 3.7. If I is an ideal of binary algebra X, then $(X/I; \circ, [0]_I)$ is a binary algebra. Moreover, the set X/I is called the *quotient binary algebra.*

Proof. Let $[x]_I, [y]_I, [z]_I \in X/I$. Then $(([x]_I \circ [y]_I) \circ ([x]_I \circ [z]_I)) \circ ([z]_I \circ [y]_I) = ([x*y]_I \circ [x*z]_I) \circ [z*y]_I = ([x*y]_I * [x*z]_I) \circ [z*y]_I = [((x*y)*(x*z))*(z*y)]_I = [0]_I$. It is clear that $[x]_I \circ [x]_I = [0]_I$ and $[x]_I \circ [0]_I = [x]_I$. Now, let $[x]_I \circ [y]_I = [0]_I$ and $[y]_I \circ [x]_I = [0]_I$. It follows that $x*y \sim 0$ and $y*x \sim 0$, and so $(x*y)*0, (y*x)*0 \in I$. Since I is an ideal of X, we get that $x*y, y*x \in I$. Consequently, $x \sim y$, proving that $[x]_I = [y]_I$. Finally, we see that $([x]_I \circ [y]_I) \circ [z]_I = [(x*y)*z]_I = [(x*z)*y]_I = ([x]_I \circ [z]_I) \circ [y]_I$. Therefore, $(X/I; \circ, [0]_I)$ is a binary algebra.

Example 3.8. According to Example 2.5, we can get that $X/I = \{[0]_I, [2]_I\}$, where $[0]_I = [1]_I = \{0, 1\}$ and $[2]_I, [3]_I = \{2, 3\}$. Let $\circ$ be defined on X/I by

$\circ$	$[0]_I$	$[2]_I$
$[0]_I$	$[0]_I$	$[2]_I$
$[2]_I$	$[2]_I$	$[0]_I$

Then $(X/I; \circ, [0]_I)$ is a binary algebra.

Lemma 3.9. Let X be a binary algebra and I, J be any sets such that $I \subseteq J \subseteq X$. Suppose that I is an ideal of X, then J is an ideal of X if and only if J/I is an ideal of X/I.

Proof. Let I be an ideal of X with $I \subseteq J \subseteq X$. Suppose firstly that J is an ideal of X, then $J/I = \{[x]_I : x \in J\}$, where $[x]_I = \{y \in J : x \sim y\}$, and $X/I = \{[x]_I : x \in X\}$, where $[x]_I = \{y \in X : x \sim y\}$. Obviously, $J/I \subseteq X/I$ and $[0]_I \in J/I$. Now, let $[x]_I \circ [y]_I \in J/I$ and $[y]_I \in J/I$. Then $[x * y]_I = [x]_I \circ [y]_I \in J/I$, it follows that $x * y \in J$ and $y \in J$. By assumption, $x \in J$. Accordingly, $[x]_I \in J/I$, this shows that J/I is an ideal of X/I.

On the other hand, suppose that J/I is an ideal of X/I and I is an ideal of X with $I \subseteq J \subseteq X$. Thus, $0 \in J$. Let $x * y \in J$ and $y \in J$. It follows that $[x * y]_I, [y]_I \in J/I$. Since $[x * y]_I = [x]_I \circ [y]_I$, so $[x]_I \circ [y]_I \in J/I$. By hypothesis, $[x]_I \in J/I$ implies $x \in J$, proving our Lemma.

Lemma 3.10. Let X be a binary algebra and I, J be any sets such that $I \subseteq J \subseteq X$. Suppose that I is a closed ideal of X. Then J is a closed ideal of X if and only if J/I is a closed ideal of X/I.

Proof. Similar to the proof of Lemma 3.9.

Lemma 3.11. Let I, J be ideals of a binary algebra X with $I \subseteq J$, then I is an ideal of X.

Proof. Obvious.

Next, the basic properties of equivalence classes are considered as the following Theorem.

Theorem 3.12. Let I be a closed ideal of a binary algebra X and $a, b \in X$. Then

(1) $[a]_I = I$ iff $a \in I$;

(2) $[a]_I = [b]_I$ or $[a]_I \cap [b]_I = \emptyset$.

Proof. Let I be a closed ideal of X and $a, b \in X$.

(1) It is clear due to the fact that $a \sim a$ for all $a \in X$ and $a * a = 0 \in I$, so we get that $a \in [a]_I = I$. Conversely, let $x \in [a]_I$. Then $x \sim a$, it follows that $x * a, a * x \in I$. By hypothesis, $x \in I$. Hence, $[a]_I \subseteq I$. To show that $I \subseteq [a]_I$, choose $x \in I$. Since I is a closed of X, we have $x * a, a * x \in I$. Thus, $x \sim a$, this means that $x \in [a]_I$ and shows that $I \subseteq [a]_I$. Consequently, $[a]_I = I$.

(2) Assume that $[a]_I \cap [b]_I \neq \emptyset$. Then there is $x \in [a]_I \cap [b]_I$ such that $x \in [a]_I$ and $y \in [a]_I$. It follows that $x \sim a$ and $x \sim b$, so $a \sim b$ by the symmetric and transitive properties. Thus $[a]_I = [b]_I$.

Theorem 3.13. If I is a closed ideal of a binary algebra X and $y \in X$, then $[y]_I$ is closed ideal of X.

Proof. Let I be a closed ideal of X and $y \in I$. It is clear that $0 \in [y]_I$. Now, suppose that $a * b \in [y]_I$ and $b \in [y]_I$. We will show that $a \in [y]_I$. Then $a * b \sim y$ and $b \sim y$, it follows that $(a * b) * y \in I$ and $b * y \in I$. By assumption, $b \in I$. From B-5, $(a * y) * b = (a * b) * y \in I$, and I is a closed ideal of X and $b \in I$; therefore, $a * y \in I$. By properties of X, we get that $((y*a)*(y*(a*b)))*(0*b) = ((y*a)*(y*(a*b)))*((a*a)*b) = ((y*a)*(y*(a*b)))*((a*b)*a) = 0$. By hypothesis, $((y * a) * (y * (a * b))) * (0 * a) \in I$, and I is closed and $b \in I$, then $0 * b \in I$. Thus $(y * a) * (y * (a * b)) \in I$. From $a * b \sim y$ and I is closed, then $y * a \in I$. Hence, $a \sim y$, this means $a \in [y]_I$. Accordingly, $[y]_I$ is an ideal of X.

Finally, let $a, b \in [y]_I$. Then $a \sim y$ and $y \sim y$, by Lemma 3.3, $a * y \sim y * y$. By B-2, it follows that $a * b \sim 0$. Thus $a * b \in [0]_I$. Now, we have $0, y \in I$ and I is closed, so $0 * y \in I$ and $y * 0 \in I$. That is, $0 \sim y$. Hence, $[0]_I = [y]_I$. By transitive, $a * b \in [y]_I$, proving our Theorem.

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Representation formulae for Bertrand curves in the Minkowski 3-space

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Abstract In this paper, we study the representation formulae for Bertrand curves in the Minkowski 3-space.

Keywords Bertrand curve, Minkowski space, represantation formulae.

§1. Introduction

Bertrand curves are one of the important and interesting topic of classical spatial curve theory [4, 6, 10]. A Bertand curve is defined as a spatial curve which shares its principal normals with another spatial curve (called *Bertrand mate*). Note that Bertrand mates are particular examples of offset curves used in computer-aided design CADG, (see [5]).

Bertrand curves are characterised as spatial curves whose curvature and torsion are in linear relation. Thus Bertrand curves may be regraded as one-dimensional analogue of Weingarten surfaces [9]. Application of Weingarten surfaces to CADG, (see [8]).

Bertrand curves and their geodesic imbedding in surfaces are recently rediscovered and studied in the context of modern soliton theory by Schief [7].

Straightforward modification of classical theory to spacelike or timelike curves in Minkowski 3-space is easily obtained, (see [1]). Null Bertrand curves in Minkowski 3-space are studied in [2]. Nonnull Bertrand curves in the n-dimensional Lorentzian space are examined in [3].

However in [1]-[2], representation formulae for Bertrand curves are not obtained.

In this paper, we study representation formulae for Bertrand curves in Minkowski 3-space.

§2. Bertrand curves and Representation Formulae in Minkowski 3-space

In this section, we collect classical results on Bertrand curves in Minkowski 3-space $\mathbb{E}_1^3$.

Let $\mathbb{E}_1^3$ be the Minkowski 3-space and γ a regular non-null curve. Then γ can be parametrised by the unit speed parameter s;

$$\langle \gamma'(s), \gamma'(s) \rangle = \varepsilon_1 = \pm 1.$$

If $\gamma(s)$ is spacelike (resp. timelike), s is called the *arclength parameter* (resp.*proper time parameter*). Let us denote by T the tangent vector field of γ;

$$T(s) := \gamma'(s).$$

Hereafter, in case $\varepsilon_1 = 1$ (spacelike curve), we assume that the acceleration vector field T' is nonnull. Then there exist vector fields N and B along γ such that

$$T' = \varepsilon_2 \kappa N, \quad N' = -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \quad B' = \varepsilon_2 \tau N. \tag{1}$$

Here ε_2 and ε_3 are *second* and *third causal characters* of γ defined by

$$\varepsilon_2 = \langle N, N \rangle, \ \varepsilon_3 = \langle B, B \rangle.$$

The vector field N and B are called the *principal normal* and *binormal vector field* of γ respectively. The functions κ and τ are called the *curvature* and *torsion* of γ respectively.

If there exists a spatial curve $\bar{\gamma}(\bar{s})$ whose principal normal direction coincides with that of original curve, then γ is said to be a *Bertrand curve*. The pair $(\gamma, \bar{\gamma})$ is said to be a *Bertrand mate*.

There are several possibilities for Bertrand mates denoted by$\{\ \overline{\varepsilon}_i\}$, the causal characters of the Bertrand mate $\bar{\gamma}$. Then by definition, $\overline{\varepsilon}_2 = \varepsilon_2$.

1. γ is spacelike with $\varepsilon_2 = 1$. In this case there are two subcases.

(a) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (+1, +1, -1)$: In this case, the mate is also spacelike. Both the rectifying planes of γ and $\bar{\gamma}$ are timelike. Thus the tangent vector fields are related by

$$\bar{T} = \pm(\cosh\theta T + \sinh\theta B)$$

for some function $\theta = \theta(s)$.

(b) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (-1, +1, +1)$: In this case, the mate is timelike. Both the rectifying planes of γ and $\bar{\gamma}$ are timelike. Thus the tangent vector fields are related by

$$\bar{T} = \pm(\sinh\theta T + \cosh\theta B)$$

for some function $\theta = \theta(s)$.

2. γ is spacelike with $\varepsilon_2 = -1$. Then $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (+1, -1, +1)$. Both the rectifying planes are spacelike. Thus

$$\bar{T} = \cos\theta T + \sin\theta B$$

for some function $\theta = \theta(s)$.

3. γ is timelike. In this case there are two subcases.

(a) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (+1, +1, -1)$: In this case, the mate is timelike. The tangent vector fields are related by

$$\bar{T} = \pm(\sinh\theta T + \cosh\theta B)$$

for some function $\theta = \theta(s)$.

(b) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (-1, +1, +1)$: In this case, the mate is timelike. The tangent vector fields are related by

$$\bar{T} = \pm(\cosh\theta T + \sinh\theta B)$$

for some function $\theta = \theta(s)$.

One can see that the case 3 is reduced to case 1. Thus we may restrict our study to case 1 and case 2.

Now let consider **case 1-(a)**:

Let $(\gamma, \bar{\gamma})$ be a Bertrand mate, then

$$\bar{\gamma}(\bar{s}) = \gamma(s) + u(s)N(s) \tag{2}$$

for some function $u(s) \neq 0$. Differentiating this, we get

$$\bar{T}(\bar{s})\frac{d\bar{s}}{ds} = (1 - \varepsilon_1 u(s)\kappa(s))T(s) + u'(s)\, N(s) + \varepsilon_3 u(s)\tau(s)\, B(s). \tag{3}$$

Since $\bar{T} \perp N$,

$$\langle \bar{T}, N\rangle \bar{s}_s = u' = 0.$$

Hence u is a nonzero constant. Denote by θ the angle between γ and $\bar{\gamma}$:

$$\bar{T} = \varepsilon(\cosh\theta\, T + \sinh\theta\, B),\ \ \varepsilon = \pm 1 \tag{4}$$

Then computing the inner product of (3) and (4), we have

$$\frac{1 - \varepsilon_1 u\kappa}{\cosh\theta} = \frac{\varepsilon_3 u\tau}{\sinh\theta} = \frac{d\bar{s}}{ds}. \tag{5}$$

Differentiating (4),

$$\varepsilon_2\bar{\kappa}\bar{s}_s\bar{N} = (\varepsilon\varepsilon_2\kappa\cosh\theta + \varepsilon\varepsilon_2\tau\sinh\theta)N + \varepsilon\theta'(\sinh\theta T + \cosh\theta B).$$

By the assumption,

$$\bar{N} = \pm N.$$

Hence

$$\theta' = 0,\ \ \bar{\kappa}\bar{s}_s = \varepsilon(\kappa\cosh\theta + \tau\sinh\theta).$$

Thus θ is a constant. If $\sinh\theta = 0$, then from (5), $\tau = 0$. In this case, γ is a planar curve. Note that planar curves are Bertrand curves. In fact, planar curves together with their parallel curves are Bertrand mates.

Next, if $\sin\theta \neq 0$, then (5) is written in the form:

$$a\kappa + b\tau = 1, \tag{6}$$

for constants a and b.

Conversely, if a spatial curve γ satisfies (6), then define $\bar{\gamma}$ by (2). Then

$$\bar{T} = \varepsilon(\cosh\theta T + \sinh\theta B).$$

Differentiating this by s, we obtain

$$\bar{\kappa}\bar{N}\bar{s}_s = \varepsilon(\kappa\cosh\theta + \tau\sinh\theta)N.$$

Hence γ is a Bertrand curve.

Thus we obatin the following result:

Theorem 1. A spatial curve is a Bertand curve in Minkowski 3-space $\mathbb{E}_1^3$ if and only if its curvature and torsion satisfy $a\kappa + b\tau = 1$ for some constants a and b.

Theorem 2. Let $(\gamma, \bar{\gamma})$ be a Bertrand mate in Minkowski 3-space $\mathbb{E}_1^3$. Then $\tau(s)\bar{\tau}(\bar{s})$ is a constant.

Proof of Theorem 2. From (2)–(6),

$$\tau = -\frac{\sinh\theta}{u}\frac{d\bar{s}}{ds}, \quad \bar{\tau} = -\frac{\sinh\bar{\theta}}{\bar{u}}\frac{ds}{d\bar{s}}, \quad \bar{u} = \pm u.$$

Hence

$$\tau\bar{\tau} = \frac{\sinh\theta\sinh\bar{\theta}}{u\bar{u}} = \text{constant}.$$

Corollary 1. Let γ be a Bertrand curve with $a\kappa + b\tau = 1$ and $\bar{\gamma}$ a Bertrand mate. Then the fundamental quantities of the Bertrand mate are given by

$$\bar{T} = \varepsilon\frac{-bT + aB}{\sqrt{b^2 - a^2}}, \quad \bar{N} = \pm N, \quad \bar{B} = \varepsilon\frac{aT - bB}{\sqrt{b^2 - a^2}},$$

$$\bar{\kappa} = -\varepsilon\frac{(-b\kappa + a\tau)}{(b^2 - a^2)\tau}, \quad \bar{\tau} = -\varepsilon\frac{a\kappa - b\tau}{(b^2 - a^2)\tau}, \quad d\bar{s} = -\sqrt{b^2 - a^2}\tau(s)ds.$$

Similarly, if we consider **case 1-(b)** we have following :

Theorem 3. A spatial curve is a Bertand curve in Minkowski 3-space $\mathbb{E}_1^3$ if and only if its curvature and torsion satisfy $a\kappa + b\tau = 1$ for some constants a and b.

Theorem 4. Let $(\gamma, \bar{\gamma})$ be a Bertrand mate in Minkowski 3-space $\mathbb{E}_1^3$. Then $\tau(s)\bar{\tau}(\bar{s})$ is a constant.

Corollary 2. Let γ be a Bertrand curve with $a\kappa + b\tau = 1$ and $\bar{\gamma}$ a Bertrand mate. Then the fundamental quantities of the Bertrand mate are given by

$$\bar{T} = \varepsilon\frac{-bT + aB}{\sqrt{a^2 - b^2}}, \quad \bar{N} = \pm N, \quad \bar{B} = \varepsilon\frac{aT - bB}{\sqrt{a^2 - b^2}},$$

$$\bar{\kappa} = -\varepsilon\frac{(-b\kappa + a\tau)}{(a^2 - b^2)\tau}, \quad \bar{\tau} = -\varepsilon\frac{a\kappa - b\tau}{(a^2 - b^2)\tau}, \quad d\bar{s} = -\sqrt{a^2 - b^2}\tau(s)ds.$$

In **case 2**, We obtained the classical results in Euclidean space.

Lemma 1. Let $e(t)$ be a unit vector field which is not parallel to a fixed plane. Take a nonzero constant a. Then

$$\alpha(t) := -\varepsilon_2 a\int e(t) \times \dot{e}(t)dt$$

is a spatial curve of constant torsion $-\varepsilon_2/a$ and binormal vector field $\pm e(t)$.

Proof of Lemma 1. Direct computations show

$$\dot{\alpha} \times \ddot{\alpha} = a^2((e \times \dot{e}) \times (e \times \ddot{e})) = a^2\{\det(e, e, \ddot{e})\dot{e} - \det(\dot{e}, e, \ddot{e})e\} = a^2\det(e, \dot{e}, \ddot{e})e.$$

Here we used the following formula in Minkowski 3-space $\mathbb{E}_1^3$:

$$(x \times y) \times (z \times w) = \det(x, z, w)y - \det(y, z, w)x$$

By the assumption, $\det(e, \dot{e}, \ddot{e}) \neq 0$, the binormal vector field of α is $B_\alpha = \pm e$.

Next, since $\det(\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}) = -\varepsilon_2 a^3 \det(e, \dot{e}, \ddot{e})^2$. Hence the torsion of α is

$$\tau_\alpha = \frac{\det(\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha})}{|\dot{\alpha} \times \ddot{\alpha}|^2} = \frac{-\varepsilon_2 a^3 \det(e, \dot{e}, \ddot{e})^2}{a^4 \det(e, \dot{e}, \ddot{e})^2} = \frac{-\varepsilon_2}{a}.$$

Conversely, let $\alpha(s)$ be a curve of constant torsion $-\varepsilon_2/a$. Here s is the arclength parameter. Then put $e(s) = B(s)$. Then the Frenet-Serret formula implies

$$e \times \dot{e} = \varepsilon_2 \tau\, B \times N = \frac{-\varepsilon_2}{a} \alpha'.$$

Hence $\alpha(s) = -\varepsilon_2 a \int^s e \times \dot{e} ds$.

Lemma 2. If a spatial curve α is of constant nonzero torsion τ_α, then the curve

$$\beta(s) = -\frac{1}{\tau_\alpha} N(s) - \varepsilon_3 \int B(s) ds$$

has constant curvature $|\tau_\alpha|$.

Proof of Lemma 2. We use the subscript $\cdot_\alpha$ for expressing geometric objects of α. By the Frenet-Serret formula for α, we have

$$\frac{d\beta}{ds} = -\frac{1}{\tau_\alpha}(-\varepsilon_1 \kappa_\alpha T_\alpha - \varepsilon_3 \tau_\alpha B_\alpha) - \varepsilon_3 B_\alpha = \varepsilon_1 \frac{\kappa_\alpha}{\tau_\alpha} T_\alpha.$$

Hence the unit tangent vectot field of β is $T_\beta = \varepsilon T$, $\varepsilon = \mathrm{sgn}(\tau)$. Hence the arclength parameter s_β of β is

$$s_\beta = \int^s \varepsilon_1 \frac{\kappa_\alpha}{|\tau_\alpha|} ds.$$

Thus

$$\frac{dT_\beta}{ds_\beta} = \varepsilon \frac{dT}{ds} \frac{ds}{ds_\beta} = \varepsilon \varepsilon_1 \varepsilon_2\, |\tau_\alpha| N, \quad N_\beta = \varepsilon N, \quad \kappa_\beta = |\tau_\alpha| = \text{constant}.$$

Lemma 3. If a spatial curve α is of constant nonzero torsion τ_α, then the curve

$$\beta(s) = a\alpha(s) + b\left(-\frac{1}{\tau_\alpha} N(s) - \varepsilon_3 \int B(s) ds\right)$$

is a Bertand curve.

Proof of Lemma 3. Direct computations show that

$$\beta' = (a + \frac{\varepsilon_1 b \kappa_\alpha}{\tau_\alpha}) T, \quad \beta'' = \frac{\varepsilon_1 b \kappa'_\alpha}{\tau_\alpha} T + \left(a \varepsilon_2 \kappa_\alpha + \frac{\varepsilon_1 \varepsilon_2 b \kappa_\alpha^2}{\tau_\alpha}\right) N,$$

$$\begin{aligned}\beta''' &= (b\varepsilon_1 \frac{\kappa''_\alpha}{\tau_\alpha} - a\varepsilon_1 \varepsilon_2 \kappa_\alpha^2 - b\varepsilon_2 \frac{\kappa_\alpha^3}{\tau_\alpha}) T + (a\varepsilon_2 \kappa'_\alpha + \frac{3\varepsilon_1 \varepsilon_2 b \kappa'_\alpha \kappa_\alpha}{\tau_\alpha}) N \\ &\quad + \left(-a\varepsilon_2 \varepsilon_3 \kappa_\alpha \tau_\alpha - b\varepsilon_1 \varepsilon_2 \varepsilon_3 \kappa_\alpha^2\right) B.\end{aligned}$$

From these

$$\kappa_\beta = \frac{|\beta' \times \beta''|}{|\beta'|^{3/2}} = \frac{-\varepsilon_2 \kappa_\alpha}{\left|a + \varepsilon_1 b \frac{\kappa_\alpha}{\tau_\alpha}\right|},$$

$$\tau_\beta = \frac{\det(\beta', \beta'', \beta''')}{|\beta' \times \beta''|^2} = \frac{-\varepsilon_3 \tau_\alpha}{a + \varepsilon_1 b \frac{\kappa_\alpha}{\tau_\alpha}}.$$

Put $\varepsilon = \mathrm{sgn}\{\varepsilon_1\varepsilon_2\varepsilon_3(a + (b\varepsilon_1\kappa_\alpha/\tau_\alpha)\}$. Then $\varepsilon b\kappa_\beta + a\tau_\beta = -\varepsilon_3\tau_\alpha = \text{constant}$.

From these Lemma, one can deduce the following:

Theorem 5. (Representation formula) Let $u(\sigma)$ be a curve in the H^2 parametrised by arclength. Then define three spatial curves α, β and γ by

$$\alpha := a \int u(\sigma) d\sigma, \quad \beta := a \tanh\theta \int u(\sigma) \times du,$$

$$\gamma := \alpha - \beta.$$

Then α is a constant curvature curve, β is a constant torsion curve and γ is a Bertrand curve. Conversely, every Bertrand curve can be represented in this form.

Proof of Theorem 5. Here we give a detailed proof.

Let $u = u(\sigma)$ be a timelike curve in H^2 parametrised by the arclength σ. Then $\{\xi = u', \eta = u \times u', u\}$ is a positive orthonormal frame field along u. Hence,

$$u'' = u + \lambda\eta, \quad u \times u'' = -\lambda\xi.$$

for some function λ. From the definition of γ, we get

$$\gamma' = a(u + \tanh\theta\eta), \quad \gamma'' = a(1 - \lambda\tanh\theta)\xi, \ \gamma''' = a(1 - \lambda\tanh\theta)(u + \lambda\eta)$$

The arclength parameter s of γ is determined by

$$\langle\gamma', \gamma'\rangle = -\frac{a^2}{\cosh^2\theta}\left(\frac{ds}{d\sigma}\right)^2.$$

Moreover we have

$$\gamma' \times \gamma'' = a^2(1 - \lambda\tanh\theta)(\eta + \tanh\theta u),$$

$$\det(\gamma', \gamma'', \gamma''') = -\langle\gamma' \times \gamma'', \gamma'''\rangle = a^3(1 - \lambda\tanh\theta)^2(\tanh\theta - \lambda).$$

Using these,

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{(1 - \lambda\tanh\theta)\cosh^2\theta}{a},$$

$$\tau = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2} = \frac{\cosh^2\theta(\tanh\theta - \lambda)}{a}.$$

Hence we have

$$a(\kappa - \tanh\theta\tau) = 1.$$

Thus γ is a Bertrand curve.

Next, we compute the curvature of α and torsion of β.

Direct computation shows that

$$\alpha' = au, \quad \alpha'' = a\xi, \quad \alpha' \times \alpha'' = a^2\eta,$$

$$\beta' = a\tanh\eta, \ \beta'' = -a\lambda\tanh\xi, \ \beta' \times \beta'' = -a^2\lambda\tanh^2\theta u,$$

$$\det(\beta', \beta'', \beta''') = a^3\lambda^2 \tanh^3\theta.$$

Hence

$$\kappa_\alpha = \frac{1}{a}, \quad \tau_\beta = \frac{1}{a\tanh\theta}.$$

Conversely. let $\gamma(s)$ be a timelike Bertrand curve with relation:

$$a(\kappa - \tanh\theta\tau) = 1.$$

Denote by σ the arclength parameter of the spherical curve:

$$u = \cosh\theta T - \sinh\theta B.$$

Then

$$u_\sigma = \frac{\cosh\theta}{a}N.$$

Hence $d\sigma/ds = \cosh\theta/a$. Thus

$$au\sigma_s = \cosh\theta(\cosh\theta T - \sinh\theta B),$$

$$a\tanh\theta u \times u_\sigma\sigma_s = -\sinh\theta(\cosh\theta B - \sinh\theta T).$$

Henceforth,

$$a\int ud\sigma - a\tanh\theta\int u \times \frac{du}{d\sigma}d\sigma = \int T(s)ds = \gamma(s).$$

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Conclusion

In this paper, we gave some characterizations for Bertrand curves and spatial curves in Minkowski 3-space We obtained representation formulae for Bertrand curves in $\mathbb{E}_1^3$. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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Smarandache's Cevians theorem (II)

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Abstract In this paper, we present the Smarandache's Cevians theorem (II) in the geometry of the triangle.

Keywords Smarandache's Cevians theorem, geometry, triangle.

§1. The main result

Smarandache's Cevians Theorem (II)
In a triangle $\triangle ABC$ we draw the Cevians AA_1, BB_1, CC_1 that intersect in P. Then:

$$\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A}.$$

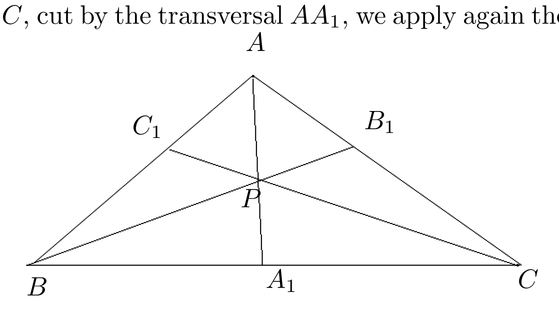
Solution 6.
In the triangle $\triangle ABC$ we apply the Ceva's theorem:

$$AC_1 \cdot BA_1 \cdot CB_1 = -AB_1 \cdot CA_1 \cdot BC_1. \tag{1}$$

In the triangle $\triangle AA_1B$, cut by the transversal CC_1, we'll apply the Menelaus' theorem:

$$AC_1 \cdot BC \cdot A_1P = AP \cdot A_1C \cdot BC_1. \tag{2}$$

In the triangle $\triangle BB_1C$, cut by the transversal AA_1, we apply again the Menelaus' theorem:



$$BA_1 \cdot CA \cdot B_1P = BP \cdot B_1A \cdot CA_1. \tag{3}$$

We apply one more time the Menelaus' theorem in the triangle $\triangle CC_1A$ cut by the transversal BB_1:

$$AB \cdot C_1P \cdot CB_1 = AB_1 \cdot CP \cdot C_1B. \tag{4}$$

We divide each relation (2), (3), and (4) by relation (1), and we obtain:

$$\frac{PA}{PA_1} = \frac{BC}{BA_1} \cdot \frac{B_1A}{B_1C}. \tag{5}$$

$$\frac{PB}{PB_1} = \frac{CA}{CB_1} \cdot \frac{C_1B}{C_1A}. \tag{6}$$

$$\frac{PC}{PC_1} = \frac{AB}{AC_1} \cdot \frac{A_1C}{A_1B}. \tag{7}$$

Multiplying (5) by (6) and by (7), we have:

$$\frac{PA}{PA_1} \cdot \frac{PB}{PB_1} \cdot \frac{PC}{PC_1} = \frac{AB \cdot BC \cdot CA}{A_1B \cdot B_1C \cdot C_1A} \cdot \frac{AB_1 \cdot BC_1 \cdot CA_1}{A_1B \cdot B_1C \cdot C_1A}.$$

But the last fraction is equal to 1 in conformity to Ceva's theorem.

§2. Unsolved Problem related to the Smarandache's Cevians theorem (II)

Is it possible to generalize this problem for a polygon?

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Strong approach of quasi-maximum likelihood estimators in a semi-parametric regression model [1]

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Abstract This paper considers a semi-parametric regression model with fixed design points. Firstly, the estimators $\hat{\beta}$ and $\hat{g}(\cdot)$ of, respectively, β and $g(\cdot)$ are derived by using the weight function and quasi-maximum likelihood methods. Then, under proper conditions, strong consistencies of these estimators are established and strong convergence rates of the estimators $\hat{\beta}$ and $\hat{g}(\cdot)$ are obtained respectively.

Keywords Semi-parametric regression model, quasi-maximum likelihood estimator, consistency, convergence rate.

§1. Introduction

Consider a semi-parametric regression model

$$y_k = X_k^T \beta + g(t_k) + e_k, k = 1, 2, \cdots, n. \tag{1}$$

where $\beta \in R^d$ is an unknown regression parameter, $g(\cdot)$ is an unknown Borel function, $t_k \in [0,1], X_k^T \in R^d$ is a deterministic design vector, y_k is an observation data, the unobserved process $\{e_k\}$ is i.i.d., and $Ee_k = 0, Var e_k = \sigma^2 < \infty$.

The study of parametric and non-parametric estimators in the semi-parametric regression model has been marked by notable developments in recent years (example [1]-[3]). But the discussion of quasi-maximum likelihood estimators in this model has been scarcely seen. For existence of quasi-maximum likelihood estimators in the semi-parametric regression model, refer to Hu Hongchang (2006) [4]. And the weak consistency of quasi-maximum likelihood estimators in the semi-parametric regression model is derived by Hu Hongchang, Xu Kan, Chen Qin (2008) [5].

In this article, we establish strong approach of quasi-maximum likelihood estimators in the semi-parametric regression model, which enrich existing estimation theories and methods for semi-parametric regression models.

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§2. Main results

§2.1 Quasi-maximum likelihood estimators

Now, we assume that β has been known and define the estimator of $g(\cdot)$

$$\hat{g}_0(t) \triangleq \hat{g}_0(t,\beta) = \sum_{k=1}^{n} W_k(t)(y_k - X_k^T\beta). \tag{2}$$

where $W_k(t)$ is a weight function. Applying (2) to (1), we have

$$\tilde{y}_k = \tilde{X}_k^T\beta + e_k, k = 1, 2, \cdots, n. \tag{3}$$

where $\tilde{X}_k = X_k - \sum\limits_{j=1}^{n} W_j(t_k)X_j, \tilde{y}_k = y_k - \sum\limits_{j=1}^{n} W_j(t_k)y_j$. By document [6], we know that the conditional log likelihood function of $y_2, \cdots, y_n$ for y_1 is

$$\Phi \triangleq \log L_n = -\frac{1}{2}(n-1)\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{k=2}^{n} e_k^2 - \frac{1}{2}(n-1)\log(2\pi).$$

Let $\frac{\partial\Phi}{\partial\sigma^2} = 0$, $\frac{\partial\Phi}{\partial\beta} = 0$, then, we have

$$\hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{k=2}^{n} e_k^2, \ \sum_{k=2}^{n} e_k\tilde{X}_k = 0.$$

Thus, we can define the estimator of $g(\cdot)$

$$\hat{g}_n(t) = \sum_{k=1}^{n} W_k(t)(y_k - X_k^T\hat{\beta}_n). \tag{4}$$

§2.2 Basic assumption

(A) Write $S_n = \sum\limits_{k=1}^{n}(\tilde{X}_k\tilde{X}_k^T)$. S_n is a positive matrix and $\lim\limits_{n\to\infty}\max\limits_{1\le k\le n}\tilde{X}_k^T S_n^{-1}\tilde{X}_k = 0$, where n is large enough.

(B) Let $|\lambda|_{\max}(\cdot)$ be the largest absolute value of eigenvalue of symmetric matrix. For $Z_n = \frac{1}{2}\sum\limits_{k=2}^{n}(\tilde{X}_k\tilde{X}_{k-1}^T + \tilde{X}_{k-1}\tilde{X^T})$, such that $\limsup\limits_{n\to x}|\lambda|_{\max}(S_n^{-\frac{1}{2}}Z_nS_n^{-\frac{T}{2}}) < 1$.

(C) $g(t), t \in (0,1)$ satisfies 1-order Lipschitz condition.

(D) The weight function satisfies the following conditions

(i) $\max\limits_{1\le k\le n}\sum\limits_{j=1}^{n} W_k(t_j) = O(1)$;

(ii) $\sup\limits_{0\le t\le 1}\max\limits_{1\le k\le n} W_k(t) = O(n^{-4/3})$;

(iii) $\sup\limits_{0\le t\le 1}\sum\limits_{k=1}^{n} I(|t - t_n > c_n|) = O(d_n)$.

where $\lim\limits_{n\to\infty} nc_n^4\log n < \infty$, $\lim\limits_{n\to\infty} nd_n^4\log n < \infty$.

(E) Let $B_n = \sum\limits_{k=1}^{n} Var\tilde{X}_k, B_n \to \infty, B_{n+1}/B_n \to 1$.

(F) $\sup\limits_{i\ge 1}|X_i| < \infty, \underline{\lambda}_n \ge cn$, where $\underline{\lambda}_n$ is the least eigenvalue of S_n.

(G) Let $\sum_i(\beta)$ be the covariance, then $\sum_i(\beta) > 0$. And the 2-order partial derivative of every element of $\sum_i(\beta)$ exists and continues. For $\forall d \in R_d$, every element of $\sum_i(\beta)$ and its 1-order and 2-order partial derivative have bound in the random bound subset in R_d.

§2.3 Lemma and theorem

Lemma 1. Under basic assumptions of (C) and (D), we have

$$\sup_t |\sum_{k=1}^n W_k(t)g(t_k) - g(t)| = O(c_n) + O(d_n), n \to \infty.$$

Lemma 2. Under basic assumption (D)(ii), we have

$$\sup_t |\sum_{k=1}^n W_k(t)e_k| = o(1) \quad a.s..$$

Lemma 3. Let $\{\xi_n\}$ be a random vector, $\{b_n\}$ be a constant sequence, and $b_n \downarrow 0$. If for a random constant sequence $\{a_n\}, 0 < a_n \uparrow \infty$, we have $\xi_n = O(a_n b_n) \ \ a.s.$, then $\xi_n = O(b_n) \ \ a.s.$.

Lemma 4. Under condition $E|e_k|^{2+\delta} < \infty$(for some $\delta > 0$), such that

$$\sup_t |\sum_{k=1}^n W_k(t)e_k| = O(n^{-2/3}(\log n)^{1/2}).$$

Theorem 1. Under basic assumption (A)-(D), there exists $E|e_k|^2 < \infty$, we have

$$\hat{\beta}_n - \beta = o(1) \quad a.s., \tag{5}$$

$$\sup_t |\hat{g}(t) - g(t)| = o(1) \quad a.s.. \tag{6}$$

Theorem 2. Under basic assumption (A)-(G), there exists $E|e_k|^{2+\delta} < \infty$(for some $\delta > 0$), we have

$$\hat{\beta}_n - \beta = O((\log\log n)^{1/2}), \tag{7}$$

$$\sup_t |\hat{g}(t) - g(t)| = O(c_n) + O(d_n) + O(n^{-2/3}(\log n)^{1/2}). \tag{8}$$

§3. Proofs

Proof of Lemma 1. See Gao J. T.(1995) [7].

Proof of Lemma 2. We can see easily $\{W_k(t)e_k\}$ is i.i.d.. Under condition (D) (ii), we have $Var(W_k(t)e_k) \le c\sigma^2 < \infty$. Then $\sum\limits_{k=1}^n \frac{Var(W_k(t)e_k)}{k^2} \le c \sum\limits_{k=1}^n \frac{1}{k^2} < \infty, n \to \infty$.
Hence, $\{W_k(t)e_k\}$ obeys strong law of large numbers by Kollmogorov criterion.
Thus, we have shown $\sup\limits_t |\sum\limits_{k=1}^n W_k(t)e_k| \to E[\sup\limits_t |\sum\limits_{k=1}^n W_k(t)e_k|] = 0 \quad a.s..$

Proof of Lemma 3. See Yue Li, Chen Xiru (2004) [8].

Proof of Lemma 4. Write $\alpha_n = n^{-2/3}(\log n)^{1/2}$, then for $\forall\varepsilon > 0$, n is large enough, with Markov's inequality, such that

$$P\{|\sum_{k=1}^{n} W_k(t_j)e_k| \geq \frac{1}{2}\alpha_n\varepsilon\} \leq \frac{\sum\limits_{k=1}^{n}[W_k(t_j)]^2 Ee_1^2}{(\varepsilon/2)^2\alpha_n^2} \leq C \cdot \frac{\sup\limits_{t}|W_k(t_j)| \cdot n^{-4/3}}{n^{-4/3}(\log n)} < \frac{1}{2}.$$

Let $\{\tilde{e}_k, k = 1, 2, \cdots, n\}$ be dependent copy of $\{e_k, k = 1, 2, \cdots, n\}$, and $\sigma_1, \sigma_2, \cdots, \sigma_n$ do not depend on $\{\tilde{e}_k, k = 1, 2, \cdots, n\}$. If $\sigma_1, \sigma_2, \cdots, \sigma_n$ are i.i.d. and $P(\sigma_1 = 1) = P(\sigma_1 = -1) = \frac{1}{2}$, then by applying symmetrization lemma, we have

$$\begin{aligned}
&P\{\max_{1\leq j\leq n}|\sum_{k=1}^{n} W_k(t_j)e_k| \geq \alpha_n\varepsilon\} \leq 2P\{\max_{1\leq j\leq n}|\sum_{k=1}^{n} W_k(t_j)(e_k - \tilde{e}_k)| \geq \tfrac{1}{2}\alpha_n\varepsilon\}\\
&= 2P\{\max_{1\leq j\leq n}|\sum_{k=1}^{n} W_k(t_j)(e_k - \tilde{e}_k)\cdot\sigma_k| \geq \tfrac{1}{2}\alpha_n\varepsilon\} \leq 4P\{\max_{1\leq j\leq n}|\sum_{k=1}^{n} W_k(t_j)e_k\sigma_k| \geq \tfrac{1}{4}\alpha_n\varepsilon\}\\
&\leq 4n\max_{1\leq j\leq n} P\{|\sum_{k=1}^{n} W_k(t_j)e_k\sigma_k| \geq \tfrac{1}{4}\alpha_n\varepsilon\} = 4n\max_{1\leq j\leq n} P(A).
\end{aligned}$$

Writing $e(n) = (e_1, e_2, \cdots, e_n)^T$. By Hoeffding inequality, we obtain

$$\begin{aligned}
P(A|e(n)) &\leq \{2\exp[-2(\alpha_n\varepsilon/4)^2/\sum_{k=1}^{n}(2W_k(t_j)e_k)^2]\} \wedge 1\\
&= \{2\exp[-C(\log n/\sum_{k=1}^{n}(W_k(t_j))^2 e_k^2 n^{4/3}]\} \wedge 1.
\end{aligned}$$

Since

$$\sum_{k=1}^{n}(W_k(t_j))^2 e_k^2 n^{4/3} \leq \sum_{k=1}^{n} Cn^{-8/3}e_k^2 n^{4/3} = C\sum_{k=1}^{n} e_k^2 n^{-4/3} \to 0 \quad a.s..$$

Hence

$$P(A|e(n)) \leq \{2\exp(-C(\log n))\} \wedge 1.$$

Thus

$$\max_{1\leq j\leq n} P(A) \leq \{2\exp(-C(\log n))\} \wedge 1.$$

The Borel-Cantelli Lemma implies that

$$\max_{1\leq j\leq n}|\sum_{k=1}^{n} W_k(t_j e_k| = o(\alpha_n).$$

We terminate the proof with $\alpha_n = n^{-2/3}(\log n)^{1/2}$.

Proof of Theorem 1. At first, we prove (5). Let

$$\rho_n(A) = \{\beta \in R^d : |(\hat{\beta} - \beta)^T \cdot S_n^{1/2}|^2 \leq A^2, n = 2, 3, \cdots, A > 0.$$

By [5], we have

$$\hat{\beta} \in \rho_n(A). \tag{9}$$

And because of [8], we obtain

$$\sup_{\hat{\beta}\in\rho_n(A)}\max_{1\leq k\leq n}|(\hat{\beta} - \beta)^T\tilde{X}_k| \to 0, n \to \infty, \forall A > 0.$$

Therefore

$$\hat{\beta}_n - \beta = o(1) \quad a.s..$$

Next, we prove (6). It is obvious that

$$\sup_t |\hat{g}(t) - g(t)| \le \sup_t |\hat{g}_0(t, \beta) - g(t)| + \sup_t |\sum_{k=1}^{n} X_k^T (\beta - \hat{\beta}_n) W_k(t)| = K_1 + K_2. \tag{10}$$

By the Lemma 1 and Lemma 2, we have

$$K_1 \le \sup_t |\sum_{k=1}^{n} W_k(t) g(t_k) - g(t)| + \sup_t |W_k(t) e_k| = O(c_n) + O(d_n) + o(1) \quad a.s.. \tag{11}$$

Since

$$K_2 = \sup_t |\sum_{k=1}^{n} X_k^T (\beta - \hat{\beta}_n) W_k(t)| \le \sum_{i=1}^{d} \{|\hat{\beta}_i - \beta_i| \cdot \sup_t |\sum_{k=1}^{n} X_{ki} W_k(t)|\}. \tag{12}$$

Then following (5), in order to obtain

$$K_2 \to 0, \quad a.s.. \tag{13}$$

We only need to prove

$$\sup_t |\sum_{k=1}^{n} X_{ki} W_k(t)| = O(1), \quad a.s.. \tag{14}$$

Because of

$$\sup_t |\sum_{k=1}^{n} X_{ki} W_k(t)| \le \sup_t \sum_{k=1}^{n} |W_k(t)| \cdot \max_{1 \le k \le n} |X_{ki}|.$$

So, by assumption (D) and (F), we have shown (14). Following (10),(11) and (13), the proof is completed.

Proof of Theorem 2. At first, we prove (7). Let

$$\rho_n^0(A) = \{\beta \in R^d : |(\hat{\beta} - \beta)^T \cdot S_n^{1/2}|^2 \le a_n^2, n = 2, 3, \cdots, A > 0.$$

where $\{a_n\}$ is a constant sequence, and $0 < a_n \uparrow \infty, a_n = o(\log n), (\log\log n)^{1/2} = o(a_n)$. Note that (9), when n is large enough, we have $\hat{\beta}_n \in \rho_n^0$. By [9], we have $\hat{\beta}_n - \beta = O(a_n)$. Hence, following Lemma 3, we obtain

$$\hat{\beta}_n - \beta = O((\log\log n)^{1/2}).$$

Next, we prove (8), which can be followed from (10), (11), (12), (14), Lemma 1 and Lemma 4, and the proof is completed.

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On some sufficient conditions for starlikeness

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Abstract Let $\mathcal{A}$ be the class of analytic functions in the unit disk that are normalized with $f(0) = f'(0) - 1 = 0$. In this paper we study the class

$$G_\lambda = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < \lambda, z \in \mathcal{U} \right\},$$

and give sharp sufficient conditions that embed it into the class

$$S^*[A, B, b] = \left\{ f \in \mathcal{A} : 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz} \right\},$$

where $-1 \leq B < A \leq 1$, $b \neq 0$ and " $\prec$ " denotes the usual subordination.

Keywords Univalent function, starlike function, criteria, Jack lemma.

§1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ that are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized such that $f(0) = f'(0) - 1 = 0$, i.e., of the form $f(z) = z + a_2 z^2 + \cdots$. For a functions $f, g \in \mathcal{A}$ we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk $\mathcal{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$. For more detail see [2].

Next, a function $f(z) \in \mathcal{A}$ belongs to the class $S^*[A, B, b]$, where $-1 \leq B < A \leq 1$ and $b \neq 0$ is a complex number, if and only if

$$ST(b) \equiv 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{1 + Az}{1 + Bz}.$$

By specifying the values A, B and/or b we obtain the following classes:

- $S^*[1, -1, 1] \equiv S^*$ is the well-known class of starlike functions;
- $S^*[1, -1, b]$ is the class of starlike functions of complex order;
- $S^*[1, -1, 1 - \beta] \equiv S^*(\beta)$, $0 \leq \beta < 1$, is the class of starlike functions of order β;
- $S^*[1, -1, e^{-i\lambda} \cos \lambda]$, $|\lambda| < \pi/2$, is the class of λ-spirallike functions;

- $S^*[1, -1, (1-\beta)e^{-i\lambda}\cos\lambda]$, $0 \le \beta < 1$, $|\lambda| < \pi/2$, is the class of λ-spirallike functions of order β;
- $S^*[1, 0, b]$ is the class defined by $|ST(b) - 1| < 1$, $z \in \mathcal{U}$;
- $S^*[\beta, 0, b]$, $0 \le \beta < 1$, is the class defined by $|ST(b) - 1| < \beta$, $z \in \mathcal{U}$;
- $S^*[\beta, -\beta, b]$, $0 \le \beta < 1$, is the class defined by $\left|\frac{ST(b)-1}{ST(b)+1}\right| < \beta$, $z \in \mathcal{U}$;
- $S^*[1, 1/M - 1, b]$, $M > 1$, is the class defined by $|ST(b) - M| < M$, $z \in \mathcal{U}$;
- $S^*[1-2\beta, -1, b]$, $0 \le \beta < 1$, is the class defined by $\operatorname{Re} ST(b) > \beta$, $z \in \mathcal{U}$.

In this paper we study the class

$$G_\lambda = \left\{ f \in \mathcal{A} : \left| \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 \right| < \lambda, z \in \mathcal{U} \right\},$$

and give sharp sufficient conditions over λ that embed it into $S^*[A, B, b]$. Similar problem was studied by different authors for some of the classes listed above an this work unifies all these efforts and rises the problem to a higher, more general, level (for more details see [3], [4], [5], [6] and [7]).

§2. Main results and consequences

The following lemma, well known as the Jack lemma, is required in our investigation.

Lemma 1. [1] Let $\omega(z)$ be a non-constant and analytic function in the unit disk $\mathcal{U}$ with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_0 then $z_0\omega'(z_0) = k\omega(z_0)$ and $k \ge 1$.

Using the Jack lemma in a similar way as in [3], [4] and [7] we obtain the following result.

Lemma 2. Let $p(z)$ be an analytic function in the unit disk $\mathcal{U}$ with $p(0) = 1$. Also let $b \ne 0$ be a complex number and A and B be such that $-1 \le B < A \le 1$ and $|B + Ab - Bb| \le 1$. If $p(z)$ satisfies

$$\frac{bzp'(z)}{[b(p(z)-1)+1]^2} \prec \frac{(A-B)bz}{[1+(B+Ab-Bb)z]^2} \equiv h_1(z), \tag{1}$$

then

$$p(z) \prec \frac{1+Az}{1+Bz}, \tag{2}$$

This result is sharp.

Proof of Lemma 2. Define a function $\omega(z)$ by $p(z) = (1 + A\omega(z))/(1 + B\omega(z))$. Then $\omega(z)$ is analytic in $\mathcal{U}$, $\omega(0) = 0$ and

$$\frac{bzp'(z)}{[b(p(z)-1)+1]^2} = \frac{(A-B)bz\omega'(z)}{[1+(B+Ab-Bb)\omega(z)]^2}.$$

We will show that $|\omega(z)| < 1$ for all $z \in \mathcal{U}$. Indeed, assume the contrary: there exists a $z_0 \in \mathcal{U}$ such that $|\omega(z_0)| = 1$. Then by the Jack lemma $z_0\omega'(z_0) = k\omega(z_0)$, $k \ge 1$ and

$$\frac{bz_0p'(z_0)}{[b(p(z_0)-1)+1]^2} = k \cdot h_1(\omega(z_0)).$$

Further, $k \cdot h_1(\omega(z_0)) \notin k \cdot h_1(\mathcal{U})$ and $h_1(\mathcal{U}) \subseteq k \cdot h_1(\mathcal{U})$, because of $|\omega(z_0)| = 1$ and $k \geq 1$, respectively. Therefore $k \cdot h_1(\omega(z_0)) \notin h_1(\mathcal{U})$ which is a contradiction to condition (1) of this lemma and so the assumption is wrong, i.e., $|\omega(z)| < 1$ for all $z \in \mathcal{U}$.

The sharpness of the result follows from the fact that for $p(z) = (1 + Az)/(1 + Bz)$ we receive $bzp'(z)/[b(p(z) - 1) + 1]^2 = h_1(z)$.

Using Lemma 2 we obtain the following theorem.

Theorem 1. Let $f(z) \in \mathcal{A}$, $b \neq 0$ be a complex number, let A and B be such that $-1 \leq B < A \leq 1$ and let $|B + Ab - Bb| \leq 1$. If

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \frac{(A - B)bz}{[1 + (B + Ab - Bb)z]^2} \equiv h_2(z), \tag{3}$$

then $f(z) \in S^*[A, B, b]$. The result is sharp.

Proof of Theorem 1. By setting $p(z) = 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)$ we can easily verify that $p(z)$ is analytic in $\mathcal{U}$, $p(0) = 1$ and

$$\frac{bzp'(z)}{[b(p(z) - 1) + 1]^2} = \frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1.$$

So, from Lemma 2 it follows both that $f(z) \in S^*[A, B, b]$ and that the result is sharp.

Remark 1. According to the definition of subordination, the sharpness of the result of Theorem 1 means that $h_2(\mathcal{U})$ is the greatest region in the complex plane with the property that if

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \in h_2(\mathcal{U}),$$

for all $z \in \mathcal{U}$ then $f(z) \in S^*[A, B, b]$.

For $b = (1 - B)/(A - B)$ or $b = (-1 - B)/(A - B)$ in Theorem 1 we have the following.

Corollary 1. Let $f(z) \in \mathcal{A}$ and let $-1 \leq B < A \leq 1$.

(i) If

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \in \mathbb{C} \setminus \left[\frac{5 - B}{4}, \infty\right) \equiv \mathbb{C} \setminus \left\{x : x \geq \frac{5 - B}{4}\right\},$$

for all $z \in \mathcal{U}$ then $f(z) \in S^*[A, B, (1 - B)/(A - B)]$.

(ii) If $B \neq -1$ and

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \in \mathbb{C} \setminus \left[\frac{5 + B}{4}, \infty\right) \equiv \mathbb{C} \setminus \left\{x : x \geq \frac{5 + B}{4}\right\},$$

for all $z \in \mathcal{U}$ then $f(z) \in S^*[A, B, (-1 - B)/(A - B)]$.

The result is sharp.

Proof of Corollary 1. First let $s = B + Ab - Bb$. Then for the function $h_2(z)$ defined in (3) we have

$$\operatorname{Re} h_2(e^{i\theta}) = 1 + (A - B)b\frac{2s + (1 + s^2)\cos\theta}{(1 + 2s\cos\theta + s^2)^2},$$

and

$$\operatorname{Im} h_2(e^{i\theta}) = (A - B)b\frac{(1 - s^2)\sin\theta}{(1 + 2s\cos\theta + s^2)^2}.$$

For $b = (1 - B)/(A - B)$ and $b = (-1 - B)/(A - B)$ we have $s = 1$ and $s = -1$, respectively. Therefore

$$\operatorname{Re} h_2(e^{i\theta}) = \begin{cases} 1 + \frac{1-B}{2(1+\cos\theta)}, & s = 1, \\ 1 + \frac{1+B}{2(1-\cos\theta)}, & s = -1. \end{cases}$$

and

$$\operatorname{Im} h_2(e^{i\theta}) = 0,$$

for all $\theta \in [0, 2\pi]$. So

$$\min\left\{\operatorname{Re} h_2(e^{i\theta}) : \theta \in [0, 2\pi]\right\} = \begin{cases} 1 + \frac{1-B}{4} = \frac{5-B}{4}, & s = 1, \\ 1 + \frac{1+B}{4} = \frac{5+B}{4}, & s = -1. \end{cases}$$

and

$$h_2(\mathcal{U}) = \begin{cases} \mathbb{C} \setminus [(5 - B)/4, \infty)\,, & s = 1 \text{ i.e. } b = (1 - B)/(A - B), \\ \mathbb{C} \setminus [(5 + B)/4, \infty)\,, & s = -1 \text{ i.e. } b = (-1 - B)/(A - B). \end{cases}$$

Therefore, the statement of this corollary follows from Theorem 1 and the definition of subordination.

Remark 2. The sharpness of Corollary 1 means that $\mathbb{C}\setminus[(5 - B)/4, \infty)$ and $\mathbb{C}\setminus[(5 + B)/4, \infty)$ are the greatest regions in the complex plane that make implications (i) and (ii) true, respectively.

Theorem 1 leads us to the following corollary about the class G_λ.

Corollary 2. Let $b \neq 0$ be a complex number, let A and B be such that $-1 \leq B < A \leq 1$ and let $|B + Ab - Bb| \leq 1$. Then $G_\lambda \subseteq S^*[A, B, b]$ where

$$\lambda = \frac{|b|(A - B)}{(1 + |B + Ab - Bb|)^2}.$$

This result is sharp, i.e., given λ is the biggest so the inclusion holds.

Proof of Corollary 2. If $f(z) \in G_\lambda$ then $f(z) \in \mathcal{A}$ and

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} \prec 1 + \lambda z.$$

Further,

$$\min\left\{|h_2(e^{i\theta}) - 1| : \theta \in [0, 2\pi]\right\} = \frac{|b|(A - B)}{(1 + |B + Ab - Bb|)^2} = \lambda,$$

and the definition of subordination brings us to $1 + \lambda z \prec h_2(z)$. Finally, Theorem 1 implies $f(z) \in S^*[A, B, b]$. The result is sharp due to the function $f(z)$ that can be obtained from

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) = \frac{1 + Az}{1 + Bz}, \quad \text{i.e.,} \quad \frac{zf'(z)}{f(z)} = 1 + \frac{(A - B)bz}{1 + Bz}.$$

For such a function

$$\frac{1 + zf''(z)/f'(z)}{zf'(z)/f(z)} - 1 = h_2(z) - 1.$$

§3. Examples

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values A, B, b.

Example 1.

(i) $G_\lambda \subseteq S^*[A,B]$ when $\lambda = (A-B)/(1+|A|)^2$. ($b=1$ in Corollary 2);

(ii) $G_\lambda \subseteq S^*(\alpha) \equiv S^*[1,-1,1-\alpha]$, $0 \le \alpha < 1$, when

$$\lambda = \begin{cases} 1, & 0 \le \alpha \le 1/2, \\ 1/\alpha - 1, & 1/2 < \alpha < 1. \end{cases}$$

($A=1$, $B=-1$ and $b = 1-\alpha$ in Corollary 2);

(iii) $G_{1/2} \subseteq S^*(2/3)$. ($\alpha = 2/3$ in Example 1 (ii));

(iv) $G_1 \subseteq S^*(1/2)$. ($\alpha = 1/2$ in Example 1 (ii));

(v) $G_\lambda \subseteq S^*[A,0,b]$ when $\lambda = A|b|/(1+A|b|)^2$, i.e., if $f(z) \in G_\lambda$ then $|ST(b)-1| < A$, $z \in \mathcal{U}$; ($B=0$ in Corollary 2).

Remark 3. The results (i) and (iv) of Example 1 are the same as Corollary 2.6 from [7] and Corollary 1 from [4], respectively.

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On defining number of subdivided certain graph

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Abstract In a given graph $G = (V, E)$, a set of vertices S with an assignment of colors to them is said to be a defining set of the vertex coloring of G if there exists a unique extension of the colors of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number. Let $G = C_{n+1}\langle 1, 4\rangle$ be circulant graph with $V(G) = \{v_1, v_2, \cdots, v_{n+1}\}$. Let G' and G'' be graphs obtained from G by subdividing of edges v_iv_{i+1} $1 \leq i \leq n+1 \pmod{n+1}$ and all of edges of G respectively. In this note, we study the chromatic and the defining numbers of G, G' and G''.

Keywords Chromatic number, circulate graph, subdivided edge, defining number.

§1. Introduction

Throughout this paper, all graphs are finite, undirected, loopless and without multiple edges. We refer the reader to [9] for terminology in graph theory. A proper k-coloring of a graph G is a labeling $f : V(G) \to \{1, 2, \cdots, k\}$ such that adjacent vertices have different labels. The labels are colors; the vertices of one color form a color class. The chromatic number of a graph G, written $\chi(G)$, is the least k such that G has a proper k-coloring. A chromatic coloring of a graph G is a proper coloring of G using $\chi(G)$ colors.

In a given graph $G = (V, E)$, a set of vertices S with an assignment of colors to them is said to be a defining set of the vertex coloring of G if there exists a unique extension of the colors of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number denoted by $d(G, c)$. For more see [3, 5, 6, 7, 8].

The circulate graph $C_{n+1}\langle 1, m\rangle$ is the graph with vertex set $\{v_1, v_2, \cdots, v_{n+1}\}$ and edge set $\{v_iv_{i+j \ (mod \ n+1)} | i \in \{1, \cdots, n+1\}\}$ and $j \in \{1, m\}$. It is necessary for circulant graphs to be connected [2]. Theoretical properties of circulant graphs have been studied extensively and are surveyed in [1]. The problem of determining chromatic numbers of circulant graphs is related to periodic colorings of integer distance graphs [4, 5].

Let $G = C_{n+1}\langle 1, 4\rangle$ be circulant graph with $V(G) = \{v_1, v_2, \cdots, v_{n+1}\}$. Let G' and G'' be graphs obtained from G by subdividing of edges v_iv_{i+1} $1 \leq i \leq n+1 \pmod{(n+1)}$ and all of edges of G respectively.

In this note, we study the chromatic and the defining numbers of G, G' and G'' are the subjects that we are interested in to study now.

§2. Chromatic number

In this section the chromatic numbers of G, G' and G'' are fixed.

Theorem 1.

(i) $\chi(G) = 3$;

(ii) $\chi(G') = 3$;

(iii) $\chi(G'') = 2$.

Proof of Theorem 1. (i): Let $n+1 = 3m+t$ where $t \in \{1,2,3\}$ and m be an integer. If $t \in \{2,3\}$, then we color the vertices of graphs with three colors $\{a,b,c\}$ as follows:

$$f(v_j) = \begin{cases} a, & \text{if } \ j = 3i+1, \\ b, & \text{if } \ j = 3i+2, \qquad (0 \le i \le m) \\ c, & \text{if } \ j = 3i+3. \end{cases}$$

If $t = 1$ we say

$$f(v_j) = \begin{cases} a, & \text{if } \ j = 3i+1 \text{ or } j = 3m, \\ b, & \text{if } \ j = 3i+2 \text{ or } j \in \{3m-2, 3m+1\}, \qquad (0 \le i \le m-2) \\ c, & \text{if } \ j = 3i+3 \text{ or } j = 3m-1. \end{cases}$$

(ii): Let $V(G') = \{v_1, v_2, \cdots, v_{n+1}\} \cup \{u_1, u_2, \cdots, u_{n+1}\} \pmod{n+1}$ where u_i is subdivide vertex of edge $v_i v_{i+1}$. Let $n+1 = 4m+t$ where m be even and $0 \le t \le 7$. We know that $\chi(G') \ge 3$. Let $\{a,b,c\}$ be three colors. We color the vertices G' by the colors $\{a,b,c\}$.

Let $t = 0$. Then

$$f(v_j) = \begin{cases} a, & \text{if } \ j = 4i+l \ \ (0 \le \text{even } i \le m-2), \\ b, & \text{if } \ j = 4i+l \ \ (0 \le \text{odd } i \le m-1), \quad (1 \le l \le 4). \end{cases}$$

and

$$f(u_j) = c \ (1 \le j \le n+1).$$

Let $1 \le t \le 4$. Then

$$f(v_j) = \begin{cases} a, & \text{if } \ j = 4i+l \ \ (0 \le \ \text{even } i \le m-2), \\ b, & \text{if } j = 4i+l \ \ (0 \le \ \text{odd } i \le m-1), \quad (1 \le l \le 4), \\ c, & \text{if } j \in \{4m+1, \cdots 4m+t\}. \end{cases}$$

and

$$f(u_j) = \begin{cases} c, & \text{if } 1 \le j \le 4m-1, \\ a, & \text{if } j \in \{4m, \cdots, 4m+t-1\}, \\ b, & \text{if } j = 4m+t. \end{cases}$$

Let $5 \leq t \leq 7$. Then

$$f(v_j) = \begin{cases} a, & \text{if } j = 4i + l \ (0 \leq \text{even } i \leq m-2) \text{ or } j \in \{4m+1, \cdots, 4m+t-4\}, \\ b, & \text{if } j = 4i + l \ \ (0 \leq \text{odd } i \leq m-1) \text{ or } j \in \{4m+5, \cdots, 4m+t\}, \quad (1 \leq l \leq 4), \\ c, & \text{if } j \in \{4m+t-3, \cdots 4m+4\}. \end{cases}$$

and

$$f(u_j) = \begin{cases} c, & \text{if } 1 \leq j \leq 4m \text{ or } j = 4m+t, \\ a, & \text{if } j \in \{4m+t-3, \cdots, 4m+t-1\}, \\ b, & \text{if } j \in \{4m+1, \cdots, 4m+t-4\}. \end{cases}$$

(iii): Let $V(G'') = \{v_1, v_2, \cdots, v_{n+1}\} \cup \{u_1, u_2, \cdots, u_{n+1}\} \cup \{w_1, w_2, \cdots, w_{n+1}\}$ (mod $n+1$) where u_i (w_i) be a subdividing vertex of edge v_iv_{i+1} (v_iv_{i+4}). We consider the proper coloring function f with criterion: $f(v_i) = a$ and $f(u_i) = b = f(w_i)$. This implies that $\chi(G'') = 2$.

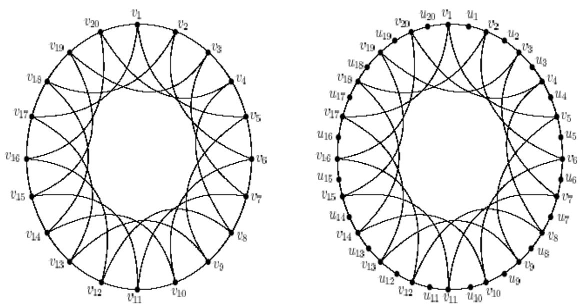

Figure 1: The Graph $G = C_{20}\langle 1, 4\rangle$ and the graph G' obtained of G

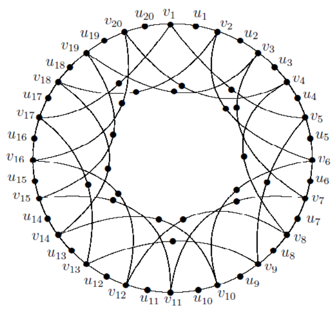

Figure 2: The Graph G'' obtained of $G = C_{20}\langle 1, 4\rangle$

§3. Defining number of vertex coloring of G'' and G

In this section we would like to verify the defining number of vertex coloring of G and G''.

Observation 1. Let H be an arbitrary graph. Then $d(H, \chi(H)) \geq \chi(H) - 1$.

In previous section we showed that $\chi(G'') = 2$, so it is straightforward that:

Observation 2. $d(G'', \chi(G'')) = 1$.

Lemma 1. $d(G, \chi(G))$ and $d(G', \chi(G'))$ are at least 3 once $n \geq 9$.

Proof of Lemma 1. Since $\chi(G) = 3 = \chi(G')$, so their defining set D has cardinality $|D| \geq 2$. On the other hand, it is easy to see by letting $D = \{x, y\}$ with an arbitrary distance, we cannot uniquely color the remaining vertices. Hence, it immediately follows that $|D| \geq 3$.

Observation 3. Let $n \geq 19$ and $D = \{v_4, v_7, v_9\}$ with $f(D) = \{3, 1, 1\}$ then set vertex $\{v_1, \cdots, v_{12}\} - \{v_4, v_7, v_9\}$ are uniquely forced to take one color.

Lemma 2. Let D be a defining set of G, then $|D| \geq \frac{n-11}{8} + 3$ vertices.

Proof of Lemma 2. Observation 3 shows that vertices of $\{v_1, v_2, \cdots, v_{12}\}$ are colored with assigned colors $\{3, 1, 1\}$ to the vertices of $\{v_4, v_7, v_9\}$. Now we show that for the remain vertices $\{v_{13}, v_{14}, \cdots, v_{n+1}\}$ in addition D must contain $\frac{n-11}{8}$ vertices. Moreover, let $m \geq 8$ which m is number of the remaining vertices. Now, we need a vertex that by it the remaining vertices get uniquely one color no more. From this, On the one hand, the farthest vertex than v_{12} (or v_1) is the vertex v_{20} (or v_{n-6}) which all vertices of v_{13} to v_{19} (or v_{n-5} to v_{n+1}) are forced to get just one color. On the other hand, for any uncolored vertex z with $d(v_{12}, z) \geq 9$, we can find at least two list coloring for uncolored vertices of v_{13} to v_{19}. Hence, it follows that $|D| \geq \frac{n-11}{8} + 3$.

We note that, however we can replace the vertex v_{17} with v_{20}, but in each of two cases all vertices v_{13} to v_{20} are forced to get a color no more and also, the vertex v_{n-3} can be replaced with v_{n-6}.

Corollary 1. Let $n - 11 = m$ then $d(G, \chi) \geq \frac{m-l}{8} + 3$ where $l \in \{0, 1, \cdots, 7\}$.

Example 1. For $n = 8$, $d(G, \chi) = 2$, for $n \in \{9, 10\}$, $d(G, \chi) = 3$ and for $n \in \{16,\ 26\}$, $d(G, \chi) \in \{3, 4, 5\}$.

Proof of Example 1. For $n = 8$, let $D = \{v_1, v_2\}$ with $f(D) = \{1, 2\}$, then it follows that $d(G, \chi) = 2$. For $n = 9$, Lemma 1 implies that $d(G, \chi) \geq 3$. The set $D_{n=9} = \{v_1, v_4, v_6\}$ with $f(D_{n=9}) = \{3, 1, 1\}$ and $D_{n=10} = \{v_1, v_6, v_7\}$ with $f(D_{n=10}) = \{2, 3, 1\}$ are a defining sets, these follow that $d(G, \chi) = 3$ where $n \in \{9, 10\}$. For $n = 16$ the set $D = \{v_4, v_7, v_9, v_{17}\}$ with $f(D) = \{3, 1, 1, 2\}$ is a defining set, thus $d(G, \chi) \leq 4$. For $n = 26$ let $D = \{v_4, v_7, v_9, v_{20}, v_{27}\}$ with $f(D) = \{3, 1, 1, 3, 1\}$, then it is a defining set. Thus $d(G, \chi) \leq 5$.

Theorem 2. Let $n \geq 11$, $n - 11 = m$ and $m = 8r + l$ where r is a nonnegative integer and $l \in \{0, 1, \cdots, 7\}$. Then $d(G, \chi) \leq \frac{m-l}{8} + 3 = r + 3$.

Proof of Theorem 2. The set vertex D with assigned colors to them is a defining set for each case. The proof is divided into the following cases by considering r and l such that $m = 8r + l$ where r is a nonnegative integer and $l \in \{0, 1, \cdots, 7\}$.

Case 1. If $l = 0$ and $r \notin \{1, 4, 7, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j + 24, \cdots\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j + 24, \cdots\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j + 24, \cdots\}. \end{cases}$$

If $l = 0$ and $r \in \{1, 4, 7, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r - 1\} \cup \{v_{n-2}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-7\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-15, n-2\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-23\}. \end{cases}$$

Case 2. If $l = 1$ and $r \in \{0, 3, 6, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-8\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-16\}. \end{cases}$$

If $l = 1$ and $r \in \{1, 4, 7, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-1\} \cup \{v_{n-7}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-8\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-16\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-24, n-7\}. \end{cases}$$

Finally, if $l = 1$ and $r \notin \{1, 4, 7, \cdots\} \cup \{0, 3, 6, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-1\} \cup \{v_{n-3}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-16\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-24, n-3\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-32, n-8\}. \end{cases}$$

Case 3. If $l = 2$ and $r = 0$ we consider $D_0 = \{v_2, v_{13}, v_{14}\}$ with $f(D_0) = \{b, c, a\}$. Now, if $r \in \{3, 6, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-3\} \cup \{v_{n-20}, v_{n-9}, v_{n-1}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-25, n-9\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-33, n-20\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-41, n-1\}. \end{cases}$$

If $l = 2$ and $r \in \{1, 4, 7, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-9\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-17\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-25, n-1\}. \end{cases}$$

If $l = 2$ and $r \in \{2, 5, 8, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-1\} \cup \{v_{n-8}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-17\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-25, n-8\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-33, n-9\}. \end{cases}$$

Case 4. If $l = 3$ and $r = 0$, we consider $D_0 = \{v_4, v_6, v_{11}\}$ with $f(D_0) = \{b, c, a\}$. Now, if $l = 3$ and $r \in \{3, 6, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-3\} \cup \{v_{n-21}, v_{n-13}, v_{n-2}\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-26\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-34,n-21,n-2\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-42,n-13\}.\end{cases}$$

If $l=3$ and $r\in\{1,4,7,\cdots\}$. Let $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r-1\}\cup\{v_{n-5}\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-10\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-18,n-5\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-26\}.\end{cases}$$

If $l=3$ and $r\in\{2,5,8,\cdots\}$. Let $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-18\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-26,n-2\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-34,n-10\}.\end{cases}$$

Case 5. If $l=4$ and $r\in\{0,3,6,\cdots\}$. For $r=0$, let $D_0=\{v_4,v_7,v_9\}$ with $f(D_0)=\{c,a,a\}$ and for $r\in\{3,6,\cdots\}$, we consider $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r-3\}\cup\{v_{n-22},v_{n-11},v_{n-4}\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-27,n-11,n-4\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-35,n-22\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-43\}.\end{cases}$$

If $l=4$ and $r\in\{1,4,7,\cdots\}$. Let $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r-1\}\cup\{v_{n-6}\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-11\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-19,n-6\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-27\}.\end{cases}$$

If $l=4$ and $r\in\{2,5,8,\cdots\}$. Let $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r-2\}\cup\{v_{n-14},v_{n-3}\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-19,n-3\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-27,n-14\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-35\}.\end{cases}$$

Case 6. If $l=5$ and $r\in\{3,6,\cdots\}$. Let $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r-3\}\cup\{v_{n-23},v_{n-12},v_{n-4}\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-28,n-12\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-36,n-23\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-44,n-4\}.\end{cases}$$

If $l=5$ and $r\in\{1,4,7,\cdots\}$. Let $D=\{v_4,v_7,v_9\}\cup\{v_{12+8t}|\ t=1,2,\cdots,r\}$ and assigned colors on D as follows:

$$f(v_i)=\begin{cases} a, & \text{where } i\in\{7,9,36,60,\cdots,j,j+24,\cdots,n-12\},\\ b, & \text{where } i\in\{28,52,\cdots,j,j+24,\cdots,n-20\},\\ c, & \text{where } i\in\{4,20,44,\cdots,j,j+24,\cdots,n-28,n-4\}.\end{cases}$$

If $l = 5$ and $r \in \{2, 5, 8, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-2\} \cup \{v_{n-15}, v_{n-7}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-20\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-28, n-15\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-36, n-7\}. \end{cases}$$

Case 7. If $l = 6$ and $r = 0$ we consider $D_0 = \{v_4, v_9, v_{14}\}$ with $f(D_0) = \{b, c, a\}$. Now, if $r \in \{3, 6, 9, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-3\} \cup \{v_{n-24}, v_{n-13}, v_{n-8}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-29, n-13\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-37, n-24, n-8\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-45\}. \end{cases}$$

If $l = 6$ and $r \in \{1, 4, 7, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-1\} \cup \{v_{n-8}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-13\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-21, n-8\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-29\}. \end{cases}$$

If $l = 6$ and $r \in \{2, 5, 8, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-21\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-29, n-5\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-37, n-13\}. \end{cases}$$

Case 8. If $l = 7$ and $r \in \{0, 3, 6, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-6\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-14\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-22\}. \end{cases}$$

If $l = 7$ and $r \in \{4, 7, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-4\} \cup \{v_{n-33}, v_{n-25}, v_{n-14}, v_{n-6}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-38, n-6\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-46, n-33, n-14\}, \\ c & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-54, n-25\}. \end{cases}$$

If $l = 7$ and $r \in \{2, 5, 8, \cdots\}$. Let $D = \{v_4, v_7, v_9\} \cup \{v_{12+8t} |\ t = 1, 2, \cdots, r-2\} \cup \{v_{n-17}, v_{n-6}\}$ and assigned colors on D as follows:

$$f(v_i) = \begin{cases} a, & \text{where } i \in \{7, 9, 36, 60, \cdots, j, j+24, \cdots, n-22, n-6\}, \\ b, & \text{where } i \in \{28, 52, \cdots, j, j+24, \cdots, n-30, n-17\}, \\ c, & \text{where } i \in \{4, 20, 44, \cdots, j, j+24, \cdots, n-38\}. \end{cases}$$

By simple verification one can see that by assigned D in each cases, the remaining vertices get uniquely one color. Hence, it is now straightforward to see that in any cases $d(G, \chi) \leq \frac{m-l}{8} + 3$.

Now Lemma 2, Example 1 and Theorem 2 yield that:

Theorem 3. Let $n \geq 11$, $n \notin \{16, 26\}$ and let $n - 11 = m$ where $m = 8r + l$, r is a nonnegative integer and $l \in \{0, 1, \cdots, 7\}$. Then $d(G, \chi) = \frac{m-l}{8} + 3$.

§4. Defining number of vertex coloring of G'

Now we discuss the defining number of G'.

Definition 1. A set of cycles of a graph is independent if the cycles are vertex disjoint, edge disjoint and there is no edge between a vertex of one cycle and a vertex of another.

Remark 1. (i): Let $n + 1 = 4k + 1 \geq 13$. Then G' with $2(n + 1)$ vertices has 1 independent cycle, namely $C_1 : v_1v_5 \cdots v_{n+1}v_4v_8 \cdots v_nv_3v_7 \cdots v_{n-1}v_2v_6 \cdots v_{n-2}v_1$ with $V(C_1) = v_1, v_5, \cdots, v_{n+1}, v_4, v_8, \cdots, v_n, v_3, v_7, \cdots, v_{n-1}, v_2, v_6, \cdots, v_{n-2}$.

(ii): Let $n + 1 = 4k \geq 12$. Then G' with $2(n + 1)$ vertices has 4 independent cycles, namely $C_1 : v_1v_5v_9 \cdots v_{n-2}v_1$, $C_2 : v_2v_6v_{10} \cdots v_{n-1}v_2$, $C_3 : v_3v_7v_{11} \cdots, v_nv_3$, and $C_4 : v_4v_8v_{12} \cdots v_{n+1}v_4$ where $V(C_1) = v_1, v_5, v_9, \cdots, v_{n-2}$, $V(C_2) = v_2, v_6, v_{10}, \cdots, v_{n-1}$, $V(C_3) = v_3, v_7, v_{11}, \cdots, v_n$ and $V(C_4) = v_4, v_8, v_{12}, \cdots, v_{n+1}$ respectively.

(iii): Let $n + 1 = 4k - 1 \geq 11$. Then G' with $2(n + 1)$ vertices has 1 independent cycle, C_1 with $V(C_1) = v_1, v_5, \cdots, v_{n-1}, v_2, v_6, \cdots, v_n, v_3, v_7, \cdots, v_{n+1}, v_4, v_8, \cdots, v_{n-2}$.

(iv): Let $n + 1 = 4k - 2 \geq 10$. Then G' with $2(n + 1)$ vertices has 2 independent cycles, C_1 and C_2 with $V(C_1) = \{v_1, v_5, v_9, \cdots, v_n, v_3, v_7, \cdots, v_{n-2}\}$ and $V(C_2) = \{v_2, v_6, \cdots, v_{n+1}, v_4, v_8, \cdots, v_{n-1}\}$ respectively.

Lemma 3. Let C be an independent cycle in G' and let v_iv_{i+4} be an arbitrary edge of C. Let f be a coloring function on G' with three colors. Then one of v_i and v_{i+4} (mod $n + 1$) is in defining set or one of u_i, u_{i-1}, u_{i+3} and u_{i+4} (mod $n + 1$) is in defining set.

Proof of Lemma 3. On the contrary assume that no of them is in defining set. Since $deg_{G'}(u_j) = 2$ and its neighbors are v_j and v_{j+1}, $(j \in \{i, i + 4\})$, then it cannot be forced to assign color. Since $deg_{G'}(v_j) = 4$ and its neighbors are v_{j-4}, v_{j+4}, u_j and u_{j-1}, it cannot also be forced to assign color. These are contradictions.

Following has straightforward proof and it is left.

Observation 4. $d(G', \chi(G')) = \begin{cases} 6, & \text{if } n + 1 = 10, \\ 8, & \text{if } n + 1 \in \{11, 13\}, \\ 10, & \text{if } n + 1 = 14. \end{cases}$

Theorem 4.

$$d(G', \chi(G')) = \begin{cases} 4(k + 1), & \text{if } n + 1 \in \{4(2k + 1), 4(2k + 1) + 2, 4(2k + 2) | k \geq 1\} \\ & \text{and } n + 1 \notin \{14 + 24l |\ l \geq 0\}, \\ 4(k + 1) + 2, & \text{if } n + 1 = 4(2k + 2) + 2 \textit{ and } k \geq 0, \\ \frac{n+1}{2}, & \text{if } n + 1 \in \{4(2k + 1) + 3,\ 4(2k + 1) + 5 |\ k \geq 1\}. \end{cases}$$

Proof of Theorem 4. Let $V(G') = \{v_1, v_2, \cdots, v_{n+1}\} \cup \{u_1, u_2, \cdots, u_{n+1}\}$ where $deg(v_i) = 4$ and $deg(u_i) = 2$. Let $4 \mid n + 1$. G' has 4 independent cycles with $\frac{n+1}{4}$ vertices $C_i = v_i, v_{i+4}, \cdots, v_{i+n-3}$ (mod $n + 1$) and $(1 \leq i \leq 4)$. Lemma 3, implies that $\lceil \frac{n+1}{8} \rceil$ vertices of each C_i are in any defining set. Hence $d(G', \chi(G')) \geq 4\lceil \frac{n+1}{8} \rceil = 4(k + 1)$ when $n + 1 \in \{4(2k + 1),\ 4(2k + 2)\}$.

Now we give a defining set of the size $4(k+1)$. Let f be a coloring function from G' to $\{a, b, c\}$. Let $D = \{v_{24t+r} |\ t \geq 0$ and $r \in \{1, 2, 3, 4, 9, 10, 11, 12, 17, 18, 19, 20\}|\ 24t + r \leq n + 1\}$. We say

$f(v_{24t+1}) = f(v_{24t+4}) = f(v_{24t+10}) = f(v_{24t+19}) = a,$
$f(v_{24t+2}) = f(v_{24t+11}) = f(v_{24t+17}) = f(v_{24t+20}) = b,$ (*)
$f(v_{24t+3}) = f(v_{24t+9}) = f(v_{24t+12}) = f(v_{24t+18}) = c.$

If $k = 1$ and $k = 2$, then $D = \{v_{24s+1}, v_{24s+2}, v_{24s+3}, v_{24s+4}, v_{24s+9}, v_{24s+10}, v_{24s+11}, v_{24s+12}\}$ and $D = \{v_{24s+1}, v_{24s+2}, v_{24s+3}, v_{24s+4}, v_{24s+9}, v_{24s+10}, v_{24s+11}, v_{24s+12}, v_{24s+17}, v_{24s+18}, v_{24s+19}, v_{24s+20}\}$ respectively. Now let $k = 3s + i$ where $i \in \{0, 1, 2\}$ and $s \geq 1$, then for $i = 0$, $D = \{v_{24t+r} |\ 0 \leq t \leq s-1\} \cup \{v_{24s+1}, v_{24s+2}, v_{24s+3}, v_{24s+4}\}$. For $i = 1$, $D = \{v_{24t+r} |\ 0 \leq t \leq s-1\} \cup \{v_{24s+1}, v_{24s+2}, v_{24s+3}, v_{24s+4}, v_{24s+9}, v_{24s+10}, v_{24s+11}, v_{24s+12}\}$. For $i = 2$, $D = \{v_{24t+r} |\ 0 \leq t \leq s-1\} \cup \{v_{24s+1}, v_{24s+2}, v_{24s+3}, v_{24s+4}, v_{24s+9}, v_{24s+10}, v_{24s+11}, v_{24s+12}, v_{24s+17}, v_{24s+18}, v_{24s+19}, v_{24s+20}\}$.

It follows that anyway D is a defining set with $|D| = 4(k+1)$ and so $d(G', \chi(G')) = 4(k+1)$ once $4 \mid n+1$.

Let $n+1 = 4(2k+1)+2$ and $n+1 \neq 14+24l$, hence it is observed easily $n+1 = 22+24l$ or $n+1 = 30+24l$. G' has 2 independent cycles with $\frac{n+1}{2}$ vertices $C_i = v_i, v_{i+4}, \cdots, v_{i+n-5}, v_{i+n-1}, v_{i+2}, \cdots, v_{i+n-7}, v_{i+n-4} \pmod{n+1}$ and $(1 \leq i \leq 2)$. Lemma 3 implies that $\lceil \frac{n+1}{4} \rceil$ vertices of each C_i are in any defining set.

Let $n+1 = 22+24l$, $(l \geq 0)$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-9}$ and assign $f(v_{n-4}) = f(v_{n-1}) = a,\ f(v_{n-3}) = b,\ f(v_{n-2}) = c.$

Let $n+1 = 30+24l$, $(l \geq 0)$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n+1}$.
It follows easily same as above $d(G', \chi(G')) = 2\lceil \frac{n+1}{4} \rceil = 4(k+1)$ once $n+1 = 4(2k+1)+2$.

Let $n+1 = 4(2k+2)+2$, hence it is observed easily $n+1 = 10+24l,\ n+1 = 18+24l$ or $n+1 = 26+24l$. G' has 2 independent cycles same above. Lemma 3 implies that $\lceil \frac{n+1}{4} \rceil 4(k+1)+2$ vertices of each C_i are in any defining set.

Let $n+1 = 10+24l$, $(l \geq 1)$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-13}$ and assign $f(v_{n-8}) = f(v_{n-5}) = c,\ f(v_{n-7}) = f(v_{n-4}) = a,\ f(v_{n-6}) = f(v_{n-3}) = b.$

Let $n+1 = 18+24l$, $(l \geq 0)$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n+1}$.

Let $n+1 = 22+24l$, $(l \geq 0)$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-9}$ and assign $f(v_{n-4}) = f(v_{n-1}) = a,\ f(v_{n-3}) = b,\ f(v_{n-2}) = c.$

Let $n+1 = 26+24l$, $(l \geq 0)$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-13}$ and assign $f(v_{n-8}) = f(v_{n-6}) = a,\ f(v_{n-7}) = f(v_{n-4}) = b,\ f(v_{n-6}) = f(v_{n-3}) = c.$

It follows easily same as above $d(G', \chi(G')) = 2\lceil \frac{n+1}{4} \rceil = 4(k+1)+2$ once $n+1 = 4(2k+1)+2$.

Let $n+1 \geq 9$ be an odd number. G' has one independent cycles with $n+1$ vertices, hence any defining set of G' has at least $\lceil \frac{n+1}{2} \rceil$ vertices.

Let $n+1 = 4(2k+1)+3,\ (k \geq 1)$. If $3 \mid n+1$ or $3 \mid n$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-2}$.

If $3 \mid n-1$ we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n1}$ and assign $f(v_{n-5}) = f(v_{n-2}) = a,\ f(v_{n-4}) = b,\ f(v_{n-3}) = c$.

Let $n+1 = 4(2k+1)+5,\ (k \geq 1)$. If $3 \mid n-4$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-12}$ and assign $f(v_{n-7}) = f(v_{n-4}) = b,\ f(v_{n-6}) = f(v_{n-3}) = c,\ f(v_{n-5}) = a$.

If $3 \mid n+1$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-4}$ and assign $f(v_{n-3}) = b$.

If $3 \mid n-6$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-4}$ and assign $f(v_{n-3}) = c$.

It follows easily same as above $d(G', \chi(G')) = \lceil \frac{n+1}{2} \rceil$ once $n+1 =\in \{4(2k+1)+3, 4(2k+1)+5\}$.

Theorem 5. Let $|V(G')| = 2(n+1)$. Then

$$d(G', \chi(G')) \leq \begin{cases} 2\lceil \frac{n+1}{4} \rceil + 2 = 4(k+1) + 2 = 10 + 12l, & \text{if } n+1 = 14 + 24l \text{ and } l \geq 1, \\ \lceil \frac{n+1}{2} \rceil + 1, & \text{if } n+1 \in \{4(2k+1)+7,\ 4(2k+1)+9 \mid k \geq 1\}. \end{cases}$$

Proof of Theorem 5. Let $n+1 = 14+24l$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots v_{n-9}$ and assign $f(v_{n-8}) = f(v_{n-6}) = f(v_{n-3}) = b,\ f(v_{n-7}) = f(v_{n-5}) = c,\ f(v_{n-4}) = a$. Hence $d(G', \chi(G')) \leq 12l + 4 + 6 = 2\lceil \frac{n+1}{4} \rceil + 2 = 4(k+1) + 2$.

Let $n+1 = 4(2k+1)+7$. If $3 \mid n$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-6}$ and assign $f(v_{n-5}) = a,\ f(v_{n-4}) = b,\ f(v_{n-3}) = c$.

If $3 \mid n-1$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-14}$ and assign $f(v_{n-9}) = f(v_{n-6}) = f(v_{n-3}) = c,\ f(v_{n-8}) = f(v_{n-5}) = a,\ f(v_{n-7}) = f(v_{n-4}) = b$.

If $3 \mid n+1$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-14}$ and assign $f(v_{n-5}) = c,\ f(v_{n-4}) = a,\ f(v_{n-3}) = b$. Hence $d(G', \chi(G')) \leq \lceil \frac{n+1}{2} \rceil + 1$.

Let $n+1 = 4(2k+1)+9$. If $3 \mid n+1$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-8}$ and assign $f(v_{n-3}) = f(v_n) = b,\ f(v_{n-2}) = c,\ f(v_{n-1}) = a$.

If $3 \mid n$, or $3 \mid n-1$, we use the assigned colors f indicated in (*) for vertices $v_1, \cdots, v_{n-8}$ and assign $f(v_{n-3}) = f(v_n) = c,\ f(v_{n-2}) = a,\ f(v_{n-1}) = b$.

Hence $d(G', \chi(G')) \leq \lceil \frac{n+1}{2} \rceil + 1$ once $n+1 \in \{4(2k+1)+7,\ 4(2k+1)+9\}$.

We close paper with open problem.

Problem 1. The bound in Theorem 5 is sharp.

Conclusion

Vertex (edge) chromatic number and in particular its defining number of many graphs have not been determined yet. In this paper we study the vertex case for a family of graphs. Since there are problems in this fields on vertex (edge) case for another family of graphs. We think that there may be found new results and it is advised to the interested researchers work on them.

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Infinite Smarandache Groupoids

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Abstract It is proved that there are infinitely many infinite Smarandache Groupoids.

Keywords Binary operation, Groupoid, semigroup, prime number, function.

§1. Introduction

The study of groupoids is very rare and meager; according to W. B. Vasantha Kandasamy, the only reason to attribute to this is that it may be due to the fact that there is no natural way by which groupoids can be constructed.

The study of Smarandache Algebraic Structures was initiated in the year 1998 by Raul Padilla following a paper written by Florentin Smarandache called "Special Algebraic Structures". In his research Padilla treated the Smarandache Algebraic Structures mainly with associative binary operation. Since then the subject has been pursued by a growing number of researchers. In [11], a systematic development of the basic non-associative algebraic structures *viz* Smarandache Groupoids, was given by W. B. Vasantha Kandasamy. Smarandache Groupoids exhibit simultaneously the properties of a semigroup and a groupoid.

In [11], most of the examples of Smarandache Groupoids, given by W. B. Vasantha Kadasamy, are finite. Further, it is said that finding Smarandache Groupoids of infinite order, seems to be a very difficult task and left as an open problem. In this papaer we give infinitely many inifinite Smarandache Groupoids by proving a theorem: "There are infinitely many Smarandache Groupoids".

In section 2 we recall some definitions, examples pertaining to groupoids, Smarandache Groupoids and integers. For basic definitions and concepts please refer [11].

§2. Preliminaries

Definition 2.1. ([11]) Given an arbitrary set P a mapping of P X P into P is called a binary operation of P. Given such a mapping $\gamma : P \text{ X } P \rightarrow P$, we use it to define a product $*$ on P by declaring $a * b = c$, if $\gamma(a, b) = c$.

Definition 2.2. ([11]) A nonempty set of elements G is said to form a groupoid if, in G, there is defined a binary operation called product denoted by $*$ such that $a * b \in G$, for all $a, b \in G$.

It is important to mention here that the binary operation $*$ defined on the set G need not be associative i.e., $(a * b) * c \neq a * (b * c)$, in general for a, b, c $\in$ G. So, we can say the groupoid $(G, *)$ is a set on which there is defined a non-associative binary operation which is closed on G. Examples of groupoids can naturally be found in the literature.

We call the order of the groupoid G to be the number of distinct elements in it. If the number of elements in G is finite we say the groupoid is of finite order or finite groupoid otherwise we say G is an infinite groupoid.

Definition 2.3. ([4]) A semigroup is a nonempty set S, in which for every ordered pair of elements $a, b \in S$ there is defined a binary operation $*$ called their product $a * b$ such that $a * b \in G$ and we have $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

Definition 2.4. ([5]) An integer n is called prime if $n > 1$ and if the only positive divisors of n are 1 and n.

Theorem 2.5. (Euclid) There are infinitely many prime numbers.

Definition 2.6. ([11]) A Smarandache Groupoid $(G, *)$ is a groupoid which has a proper subset S, $S \subset G$, such that S under the operations of G is a semigroup.

Example 2.7. Let $(G, *)$ be a groupoid given by the following table.

$*$	0	1	2	3	4	5
0	0	3	0	3	0	3
1	1	4	1	4	1	4
2	2	5	2	5	2	5
3	3	0	3	0	3	0
4	4	1	4	1	4	1
5	5	2	5	2	5	2

Table 1.

Clearly, $S_1 = \{0, 3\}$, $S_2 = \{1, 4\}$ and $S_3 = \{2, 5\}$ are proper subsets of G which are semigroups of G. So, $(G, *)$ is a Smarandache Groupoid.

Definition 2.8. Let G be a Smarandache Groupoid, if the number of elements in G is finite we say G is a finite Smarandache Groupoid otherwise, infinite Smarandache Groupoid.

Definition 2.9. Let G be a Smarandache Groupoid, G is said to be a Smarandache commutative groupoid if there is a proper subset, which is a semigroup, is a commutative semigroup.

§3. Proof of the theorem

In this section we prove our main theorem that "There are infinitely many infinite Smarandache Groupoids". Here, for any real number x, [x] means the greatest integer less than or equal to x.

Theorem 3.1. There are infinitely many infinite Smarandache Groupoids.

Proof. We construct infinitely many infinite Smarandache Groupoids in two ways.

(3.1.1). Define the operation $*$ on the set Z of integers by $x * y = [\frac{x}{p}] + [\frac{y}{p}]$, where p is a prime number, for all $x, y \in Z$. It is immediate that $*$ is a binary operation on Z as $[\frac{x}{p}], [\frac{y}{p}]$ are always integers. Next, we observe that the operation $*$ is not associative. For, take $p = 2$.

$$(2 * 3) * 4 = ([\tfrac{2}{2}] + [\tfrac{3}{2}]) * 4 = 2 * 4 = [\tfrac{2}{2}] + [\tfrac{4}{2}] = 3.$$

On the other hand,

$$2 * (3 * 4) = 2 * ([\tfrac{3}{2}] + [\tfrac{4}{2}]) = 2 * 3 = [\tfrac{2}{2}] + [\tfrac{3}{2}] = 2.$$

So, the operation $*$ is not associative. Hence, the structure $(Z, *)$ is an infinite groupoid.

Now, we show that the groupoid $(Z, *)$ is a Smarandache Groupoid. For a given prime number p, consider the proper subset $S = \{0, 1, 2, 3, \ldots, p-1\}$ of the set Z. We can easily see that $(S, *)$ is a semigroup as $a * b = 0$ for all $a, b \in S$. Further, for a given prime p , there are many proper subsets of Z, which are always semigroups under $*$. Hence, $(Z, *)$ is an infinite Smarandache Groupoid.

In view of the definition of the operation $*$ and $Theorem[2.5]$ it is evident that there are infinitely many ways to define the operation $*$ as for each prime number there is such a binary operation on Z . Hence, our assertion is established.

(3.1.2). Let Z^+ be the set of positive integers and p be a prime number. For a given $a \in Z^+$, define a function $f_a : pz^+ \to z^+$ by $f_a(x) = [\frac{x}{pa}] + 1$, for all $x \in Z^+$. Now, we define the operation $*$ on Z^+ as $x * y = f_x(y)$, for all $x, y \in Z^+$. It is obvious that $*$ is a binary operation on Z^+ as $x * y = f_x(y) = [\frac{y}{px}] + 1$ is always a positive integer, for all $x, y \in Z^+$.

Next, we observe that $*$ is not associative. For, take $p = 3$.

$$(4 * 5) * 6 = (f_4(5)) * 6 = ([\tfrac{5}{12}] + 1) * 6 = 1 * 6 = f_1(6) = [\tfrac{6}{3}] + 1 = 3.$$

On the other hand,

$$4 * (5 * 6) = 4 * (f_5(6)) = 4 * ([\tfrac{6}{15}] + 1) = 4 * 1 = f_4(1) = [\tfrac{1}{12}] + 1 = 1.$$

So, the operation $*$ is not associative. Hence, $(Z^+, *)$ is an infinite groupoid. Now, we show that $(Z^+, *)$ is a Smarandache Groupoid. For a given prime number p, consider a proper subset $S = \{1, 2, \ldots, p-1\}$ of the set Z^+. We can easily see that $(S, *)$ is a semigroup as $a * b = f_a(b) = [\frac{b}{pa}] + 1 = 1$, for all $a, b \in S$.

Further, for a given prime p there are many proper subsets of Z^+ which are semigroups. Hence, $(Z^+, *)$ is an infinite Smarandache Groupoid. Inview of the argument provided in (3.1.1), our assertion is established immediately.

Corollary 3.2. There are infinitely many infinite Smarandache commutative groupoids.

Proof. Obvious, as the binary operation $*$ defined on the set Z of integers in (3.1.1) is commutative.

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